An error estimate for the parabolic approximation of multidimensional scalar conservation laws with boundary conditions

Jerome Droniou, Cyril Imbert, Julien Vovelle

An error estimate for the parabolic approximation of multidimensional scalar conservation laws with boundary conditions

J. Droniou* C. Imbert† J. Vovelle‡

13/10/2003

Abstract

We study the parabolic approximation of a multidimensional scalar conservation law with initial and boundary conditions. We prove that the rate of convergence of the viscous approximation to the weak entropy solution is of order $\eta^{1/3}$, where η is the size of the artificial viscosity. We use a kinetic formulation and kinetic techniques for initial-boundary value problems developed by the last two authors in a previous work.

Résumé

Nous étudions l'approximation parabolique d'une loi de conservation scalaire multi-dimensionnelle avec conditions initiales et aux limites. Nous prouvons que la vitesse de convergence de l'approximation visqueuse vers la solution entropique est de l'ordre de $\eta^{1/3}$, où η est la taille de la viscosité artificielle. Nous utilisons une formulation et des techniques cinétiques développées pour des problèmes au bord par les deux derniers auteurs dans un travail précédent.

keywords: conservation law, initial-boundary value problem, error estimates, parabolic approximation, kinetic techniques.
AMS classification: 35L65, 35D99, 35F25, 35F30, 35A35

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$ with Lipschitz continuous boundary. Let $n(\overline{x})$ be the outward unit normal to Ω at a point $\overline{x} \in \partial\Omega$, $Q = (0, +\infty) \times \Omega$ and $\Sigma = (0, +\infty) \times \partial\Omega$. We consider the following multidimensional scalar conservation law

$$\partial_t u + \mathrm{div} A(u) = 0 \text{ in } Q, \tag{1a}$$

with the initial condition

$$u(0, x) = u_0(x), \forall x \in \Omega, \tag{1b}$$

and the boundary condition

$$u(s, y) = u_b(s, y), \forall (s, y) \in \Sigma. \tag{1c}$$

*Département de Mathématiques, CC 051, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France. email: `droniou@math.univ-montp2.fr`

†Département de Mathématiques, CC 051, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France. email: `imbert@math.univ-montp2.fr`

‡CMI, Université de Provence, 39, rue Joliot-Curie, 13 453 Marseille cedex 13, France. email: `vovelle@cmi.univ-mrs.fr`

It is known that entropy solutions must be considered if one wants to solve scalar conservation laws (Equation (1a) is replaced by a family of inequalities — see [8] for the Cauchy problem) and that the Dirichlet boundary conditions are to be understood in a generalized sense (see [1] for regular initial and boundary conditions and [11] for merely bounded data).
In this paper, we estimate the difference between the weak entropy solution of (1) and the smooth solution of the regularized parabolic equation

$$\partial_t v + \mathrm{div} A(v) = \eta \Delta v \text{ in } Q, \tag{2}$$

satisfying the same initial and boundary conditions. Throughout the paper, we make the following hypotheses on the data: the flux function A belongs to $\mathsf{C}^2(\mathbb{R})$, the initial condition u_0 is in $\mathsf{C}^2(\overline{\Omega})$, the boundary $\partial\Omega$ of the domain Ω is C^2, the boundary condition u_b belongs to $\mathsf{C}^2(\overline{\Sigma})$. In that case, there exists a unique solution v^η (regular outside $\{0\} \times \partial\Omega$) to the problem (2)-(1b)-(1c).

The aim of this paper is to prove the following error estimate.

Theorem 1 *Suppose that Ω is C^2, $A \in \mathsf{C}^2(\mathbb{R})$, $u_0 \in \mathsf{C}^2(\overline{\Omega})$ and $u_b \in \mathsf{C}^2(\overline{\Sigma})$. Let u be the weak entropy solution of* (1) *and let v^η be the solution of the approximate parabolic problem* (2)-(1b)-(1c)*. Let $T_0 > 0$; there exists a constant C only depending on $(\Omega, u_b, u_0, A, T_0)$ such that, for all $t \in [0, T_0]$,*

$$||u(t) - v^\eta(t)||_{\mathsf{L}^1(\Omega)} \leqslant C\eta^{1/3}\,. \tag{3}$$

We now recall what is known about error estimates for approximations of conservation laws.
In the case where the function u is smooth (a feature which, we recall, requires the data to be smooth, compatible and the time T_0 to be small enough), error estimates of order

$$\begin{cases} \eta^{1/2} & \text{if the boundary is characteristic} \\ \eta & \text{if the boundary is not characteristic} \end{cases} \tag{4}$$

in $\mathsf{L}^\infty(0, T; \mathsf{L}^1(\Omega))$ have been given (see Gues [5], Gisclon and Serre [3], Grenier and Gues [4], Joseph and LeFloch [7], Chainais-Hillairet and Grenier [2] and references therein). The technique of *boundary layer analysis* developed in those articles is devoted to the investigation of the initial-boundary value problem for *systems* of conservation laws (and not only for a single equation). Roughly speaking, the viscous approximation v^η is decomposed as $v^\eta = u + c^\eta + \text{(remainder)}$ where c^η characterizes the boundary layer which appear in the vicinity of $\partial\Omega$. Estimates on $v^\eta - u$ are then consequences of estimates on $c^\eta + \text{(remainder)}$ (see Appendix 8.1).
To our knowledge, there does not exist other techniques of analysis which would confirm the error estimate (4). On the contrary, many techniques have been set and improved to analyse the error of approximation for the *Cauchy Problem* ($\Omega = \mathbb{R}^d$) for conservation laws (and results of sharpness of error estimates have also been delivered). The first error estimate for the Cauchy problem is given by Kuznetčov in 1976 [9]: an adaptation of the proof of the result of comparison between two weak entropy solutions given by Kružkov [8] yields an error estimate of order $1/2$ in the L^1-norm. The reader interested in more precise, more general and more recent results is invited to consult the compilation made by Tang [14], the introduction of [13], and references therein.
We establish here Estimate (3) for arbitrary times T_0; in particular, the possible occurence of shocks is taken into account: u is the *weak* entropy solution to Problem (1) and has no more regularity, in general, than the ones stated in Proposition 1. As a consequence, u may be irregular in the vicinity of $\partial\Omega$ and this constitutes an obstacle to the analysis of the rate of convergence of v^η. To circumvent this obstacle, we use the kinetic formulation of [6] (an adaptation to boundary problems of the kinetic formulation introduced in [10]) and adapt the technique of error estimate developed by Perthame for the analysis of the Cauchy Problem [12]. We then obtain a rate of convergence of $1/3$. The accuracy or non-sharpeness of this order (compare to (4)) remains an open problem for us.

The paper is dedicated to the proof of Theorem 1. It is organized as follows. We begin with some preliminaries, mainly to state (or recall) the kinetic formulations of both hyperbolic and parabolic equations. In order to enlight the key ideas of this rather technical proof, we present its skeleton in Subsection 2.4. In Section 3, we obtain a first estimate in the interior of the domain; then, in Sections 4 and 5, we transport the equations so that Ω becomes a half space and we regularize them in order to use the solution of one of them as a test function in the other. Eventually, in Section 6, we conclude the proof of Theorem 1 by getting an estimate of the boundary term which appears at the end of Section 5.

2 Preliminaries

In order to clarify computations, we drop the superscript η in v^η and simply write v for the approximate solution. We prove Theorem 1 in several steps.

2.1 Known estimates on u and v

We gather in the following proposition the estimates we will need to prove Theorem 1. We refer to [1] for a proof of these results.

Proposition 1 *Assume that Ω is C^2, $A \in \mathsf{C}^2(\mathbb{R})$, $u_0 \in \mathsf{C}^2(\overline{\Omega})$ and $u_b \in \mathsf{C}^2(\overline{\Sigma})$. There exists $C > 0$ only depending on $(\Omega, u_b, u_0, A, T_0)$ such that*

1. *the functions $u, v : [0, T_0] \to \mathsf{L}^1(\Omega)$ are C-Lipschitz continuous*
2. *for all $t \in (0, T_0)$, $\int_\Omega |\partial_t v(t, \cdot)| \leqslant C$*
3. *for all $t \in [0, T_0]$, $|u(t, \cdot)|_{\mathsf{BV}(\Omega)} \leqslant C$ and $|v(t, \cdot)|_{\mathsf{BV}(\Omega)} \leqslant C$.*

2.2 Notations

Let us introduce some local charts of Ω. Since Ω is C^2 and bounded, we can find a finite cover $\{O_i\}_{i\in\{0,\ldots,n\}}$ of $\overline{\Omega}$ by open sets of $\mathbb{R}^d$ such that $\overline{O_0} \subset \Omega$ and that, for all $i \in \{1, \ldots, n\}$, there exists a C^2-diffeomorphism $h_i : O_i \to B^d$ (the unit ball in $\mathbb{R}^d$) satisfying

- $\partial\Omega \subset \cup_{i=1}^n O_i$;
- $h_i(O_i \cap \partial\Omega) = B^{d-1} := B^d \cap (\mathbb{R}^{d-1} \times \{0\})$;
- $h_i(O_i \cap \Omega) = B^d_+ := B^d \cap (\mathbb{R}^{d-1} \times (0, +\infty))$.

Let $(\lambda_i)_{i\in\{0,\ldots,n\}}$ be a partition of the unity on $\overline{\Omega}$, subordinate to the cover $\{O_i\}_{i\in\{0,\ldots,n\}}$.

In the following, when a quantity appears with a bar above, it denotes something related to the boundary of Ω (possibly transported on B^{d-1} by a chart): either a variable on $\partial\Omega$ or the value of a function on this boundary. The values of a function ϕ at $t = 0$ are denoted by $\phi^{(t=0)}$.
Here are other general notations, related to the regularization of the equations. Let $\theta \in \mathsf{C}^\infty_\mathsf{c}(]1/2, 1[; \mathbb{R}^+)$ be such that $\int_\mathbb{R} \theta = 1$ and define, for $\tau > 0$, $\theta_\tau(\cdot) = \frac{1}{\tau}\theta(\frac{\cdot}{\tau})$ (right-decentred regularizing kernel). When necessary, we define regularizing kernels ρ_μ in space (either the whole space or on the (transported) boundary of Ω) or space-time variables; when such a kernel on $\mathbb{R}^N$ ($N = d-1$, $N = d$ or $N = d+1$) is given and f is a function defined and locally integrable on a set $S \subset \mathbb{R}^N$, we denote, for $z \in \mathbb{R}^N$,

$$f^\mu(z) = \int_S f(r)\rho_\mu(z - r)\, dr\,,$$

i.e. f^μ is the convolution of ρ_μ by the extension of f by 0 outside S. We have then, for all $\phi \in \mathsf{L}^1(\mathbb{R}^N)$ with compact support,

$$\int_S f(\phi \star \check{\rho_\mu}) = \int_{\mathbb{R}^N} f^\mu \phi$$

(where $\check{\rho_\mu}(z) = \rho_\mu(-z)$).

2.3 Kinetic formulations of (1) and (2)

The function sgn_+ is defined by $\mathrm{sgn}_+(s) = 0$ if $s \leqslant 0$ and $\mathrm{sgn}_+(s) = 1$ if $s > 0$; similarly, $\mathrm{sgn}_-(s) = -1$ if $s < 0$ and $\mathrm{sgn}_-(s) = 0$ if $s \geqslant 0$. Let $D = \sup(||u_b||_\infty, ||u_0||_\infty)$.

Let us recall the kinetic formulation of (1) obtained in [6]: there exists a bounded nonnegative measure $m \in \mathcal{M}^+(Q \times \mathbb{R}_\xi)$, which has a compact support with respect to ξ, and two nonnegative measurable functions $m^b_+, m^b_- \in \mathsf{L}^\infty_{\mathrm{loc}}(\Sigma \times \mathbb{R}_\xi)$ such that the function m^b_+ vanishes for $\xi \gg 1$ (resp. the function m^b_- vanishes for $\xi \ll -1$) and such that the functions $f_\pm(t,x,\xi) = \mathrm{sgn}_\pm(u(t,x) - \xi)$ associated with u satisfy, for any $\phi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^{d+2})$

$$\int_{Q\times\mathbb{R}_\xi} f_\pm(\partial_t + a\cdot\nabla)\phi + \int_{\Omega\times\mathbb{R}_\xi} f^0_\pm \phi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi} (-a\cdot n) f^\tau_\pm \overline{\phi} = \int_{Q\times\mathbb{R}_\xi} \partial_\xi \phi dm \tag{5}$$

where $a = A'$, $f^0_\pm(x,\xi) = \mathrm{sgn}_\pm(u_0(x) - \xi)$ and $f^\tau_\pm(t,\overline{x},\xi) = \mathrm{sgn}_\pm(\overline{u}(t,\overline{x}) - \xi)$ satisfies

$$(-a\cdot n) f^\tau_\pm = M f^b_\pm + \partial_\xi m^b_\pm \tag{6}$$

with $f^b_\pm(t,\overline{x},\xi) = \mathrm{sgn}_\pm(u_b(t,\overline{x}) - \xi)$ and M the Lipschitz constant of the flux function A on $[-D,D]$. This formula is the kinetic formulation of the BLN condition (see [1]).

We next give a kinetic formulation for the approximate solution. Consider two test functions $\varphi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}_t \times \mathbb{R}^d)$, $\psi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}_\xi)$ and define $E(\alpha) = \int \psi(\xi)\mathrm{sgn}_\pm(\alpha - \xi) d\xi$ and $H(\alpha) = \int a(\xi)\psi(\xi)\mathrm{sgn}_\pm(\alpha-\xi)d\xi$. Note that $E' = \psi$ and $H' = E'a$. Now multiply the equation $\partial_t v + \mathrm{div} A(v) = \eta \triangle v$ by $\varphi(t,x)\psi(v(t,x)) = \varphi(t,x)E'(v(t,x))$, integrate over Q and integrate by part (using the fact that v is C^2 outside $\{0\}\times\partial\Omega$)

$$\begin{aligned}\int_Q E(v)\partial_t\varphi + H(v)\cdot\nabla\varphi + \int_\Omega E(u_0)\varphi^{(t=0)} - \int_\Sigma H(u_b)\cdot n\overline{\varphi}\\ = \int_Q \eta E'(v)\nabla v\cdot\nabla\varphi - \int_\Sigma \eta E'(u_b)\overline{\nabla v}\cdot n\overline{\varphi} + \int_Q \eta E''(v)|\nabla v|^2\,\varphi.\end{aligned}$$

Using the definition of E and H, we obtain, denoting $g_\pm(t,x,\xi) = \mathrm{sgn}_\pm(v(t,x) - \xi)$,

$$\int_{Q\times\mathbb{R}_\xi} g_\pm(\partial_t + a\cdot\nabla)\phi - \int_{Q\times\mathbb{R}_\xi} \eta\delta_v \nabla v\cdot\nabla\phi + \int_{\Omega\times\mathbb{R}_\xi} f^0_\pm\phi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi} \overline{G_\pm}\ \overline{\phi} = \int_{Q\times\mathbb{R}_\xi} \partial_\xi\phi dq \tag{7}$$

where $\phi(t,x,\xi) = \varphi(t,x)\psi(\xi)$ and

$$\begin{aligned}&\overline{G_\pm} = (-a\cdot n)f^b_\pm + \eta\delta_{u_b}\overline{\nabla v}\cdot n\\ &q = \eta\delta_v|\nabla v|^2 \geqslant 0\end{aligned}$$

(notice that the support of q is compact with respect to ξ). Using a classical argument relying on convolution techniques, we claim that (7) holds true for any test function $\phi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R}^{d+2})$.

Remark 1 *i) Since f_+, f^0_+, f^τ_+ and m vanish for $\xi \gg 1$, Equation* (5) *with f_+ holds true when the support of the test function ϕ is merely lower bounded (and not necesseraly compact) with respect to ξ. Similarly, we can apply* (7) *with g_- to test functions the support of which is only upper bounded with respect to ξ. Notice also that, in all the following, though we write integrals in ξ on the whole of $\mathbb{R}_\xi$, the integrands we consider are null outside a fixed compact (namely $[-D,D]$) of $\mathbb{R}_\xi$; we use this in some estimates, without recalling it.*

ii) Equations (5) *and* (7) *can be applied to certain test functions which are not fully regular but have some monotony properties with respect to ξ, provided we replace the equality by an inequality (the sign of which is given by the monotony of the test function). More precisely, we consider, in the following, test functions of the kind $\phi(t,x,\xi) = \int_0^\infty \int_\Omega \varphi(t,x,s,y)\mathrm{sgn}_\pm(W(s,y)-\xi)\,dy\,ds$, where W is bounded and φ is regular and has a fixed sign; we can approximate $\mathrm{sgn}_\pm$ by some non-decreasing and regular functions $\mathrm{sgn}_{\pm,\delta}$; then, applying* (5) *or* (7) *to $\phi_\delta(t,x,\xi) = \int_0^\infty \int_\Omega \varphi(t,x,s,y)\mathrm{sgn}_{\pm,\delta}(W(s,y)-\xi)\,dy\,ds$, which is regular and has the same monotony properties as ϕ (with respect to ξ), we notice that the right-hand side has a fixed sign; then, passing to the limit $\delta \to 0$, we see that these inequalities are satisfied with ϕ.*

2.4 Main ideas of the proof

We present here formal manipulations which enable to understand the key steps of the proof. Let $(t,x) \mapsto \varphi(t,x)$ be a non-negative regular function. Plugging $\phi = \varphi g_-$ in (5) and $\phi = \varphi f_+$ in (7), we obtain

$$\int_{Q\times\mathbb{R}_\xi} f_+(\partial_t + a\cdot\nabla)(\varphi g_-) + \int_{\Sigma\times\mathbb{R}_\xi} (-a\cdot n) f_+^\tau f_-^b \overline{\varphi} \leqslant 0$$

and

$$\int_{Q\times\mathbb{R}_\xi} g_-(\partial_t + a\cdot\nabla)(\varphi f_+) + \int_{\Sigma\times\mathbb{R}_\xi} (-a\cdot n) f_-^b f_+^\tau \overline{\varphi} - \eta\int_{Q\times\mathbb{R}_\xi} \delta_v \nabla v\cdot\nabla(f_+\varphi) + \eta\int_{\Sigma\times\mathbb{R}_\xi} \delta_{u_b} \overline{\nabla v}\cdot n\overline{f_+}\overline{\varphi} \leqslant 0$$

(since $f_+^0 f_-^0 = 0$). Summing these inequalities and integrating by parts, it comes

$$\int_{Q\times\mathbb{R}_\xi} f_+ g_-(\partial_t + a\cdot\nabla)\varphi \leqslant -\int_{\Sigma\times\mathbb{R}_\xi} (-a\cdot n) f_+^\tau f_-^b \overline{\varphi} - \eta\int_{\Sigma\times\mathbb{R}_\xi} \delta_{u_b} \overline{\nabla v}\cdot n\overline{f_+}\overline{\varphi} + \eta\int_{Q\times\mathbb{R}_\xi} \delta_v \nabla v\cdot\nabla(f_+\varphi).$$

Taking $\varphi(t,x) = \omega_\zeta(t)$ with $(\omega_\zeta)_{\zeta>0}$ which converges to the characteristic function of $[0,T]$ and $\omega'_\zeta \to -\delta_T$, this gives

$$\begin{aligned}\int_\Omega (u-v)^+(T,x)\,dx &= \int_{\Omega\times\mathbb{R}_\xi} (-f_+ g_-)^{(t=T)} \\ &\leqslant -\int_{[0,T]\times\partial\Omega\times\mathbb{R}_\xi} (-a\cdot n) f_+^\tau f_-^b - \eta\int_{[0,T]\times\partial\Omega\times\mathbb{R}_\xi} \delta_{u_b}\overline{\nabla v}\cdot n\overline{f_+} + \eta\int_{[0,T]\times\Omega\times\mathbb{R}_\xi} \delta_v \nabla v\cdot\nabla f_+. \quad (8)\end{aligned}$$

The functions f_+ and g_- are not regular enough to justify such manipulations, which are therefore performed with f_+^ε and g_-^ν, regularized versions of these applications. The smoothing of g_- is purely technical and we immediately let $\nu \to 0$; at the contrary, the way we define f_+^ε is crucial for the proof. A decentralizing regularization allows to get rid of the second term of the right-hand side of (8); the size of the regularization being ε, $||\nabla f_+^\varepsilon||_\infty$ is bounded by C/ε and the last term of (8) is of order η/ε. There remains to estimate the first term of the right-hand side of (8), which is the aim of a whole section (Section 6); the idea is to re-use the kinetic equation satisfied by v.

3 Estimate in the interior of the domain

In this section, we let $\lambda = \lambda_0$ (we drop the subscript 0) and $K := \mathrm{supp}(\lambda_0)$. In order to obtain an estimate on the interior of the domain, we need to localize using λ, regularize both kinetic equations in order to combine them, proceeding as we did when proving the Comparison Theorem in [6]. This step is more or less classical.

Let $\alpha > 0$ and $0 < \varepsilon < \mathrm{dist}(K, \partial\Omega)$; denote $\gamma_\varepsilon(x) = \prod_{i=1}^d \theta_\varepsilon(x_i)$. Taking $\phi \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^{d+2})$ with support in $\mathbb{R}\times K\times\mathbb{R}_\xi$ and using $\phi\star(\check{\gamma_\varepsilon}\otimes\check{\theta_\alpha})$ ([1]) — notice that this function is null on the boundary of Ω — as a

[1]Here and after, the tensorial product is used to recall that $\check{\gamma_\varepsilon}$ and $\check{\theta_\alpha}$ use different variables (for example, $\check{\gamma_\varepsilon}\otimes\check{\theta_\alpha}(t,x) = \check{\gamma_\varepsilon}(x)\check{\theta_\alpha}(t)$) and the convolution product $\star$ never involves the kinetic variable ξ.

test function in (5) with f_+, we find

$$\int_{\mathbb{R}^{d+2}} f_+^{\alpha,\varepsilon}(\partial_t + a\cdot\nabla)\phi + \int_{\mathbb{R}^{d+2}} f_+^{0\;\varepsilon}\otimes\theta_\alpha\phi = \int_{\mathbb{R}^{d+2}} \partial_\xi\phi\, dm^{\alpha,\varepsilon} \tag{9}$$

(where $m^{\alpha,\varepsilon}$ is the convolution in (t,x) of $\gamma_\varepsilon\otimes\theta_\alpha$ by the extention of m by 0 outside $Q\times\mathbb{R}_\xi$). We next regularize the equation satisfied by g_-, using the same method but different parameters $\beta>0$ and $0<\nu<\mathrm{dist}(K,\partial\Omega)$: we obtain for the same ϕ's

$$\int_{\mathbb{R}^{d+2}} g_-^{\beta,\nu}(\partial_t + a\cdot\nabla)\phi + \int_{\mathbb{R}^{d+2}} f_-^{0\;\nu}\otimes\theta_\beta\phi - \eta\int_{Q\times\mathbb{R}_\xi}\delta_v\nabla v\cdot(\nabla\phi)\star(\check{\gamma_\nu}\otimes\check{\theta_\beta}) = \int_{\mathbb{R}^{d+2}} \partial_\xi\phi\, dq^{\beta,\nu}. \tag{10}$$

Suppose that $\phi\in\mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^{d+1})$ is non-negative with support in $\mathbb{R}\times K$ and apply (9) to the test function $-g_-^{\beta,\nu}(t,x,\xi)\phi(t,x)$ and (10) to $-f_+^{\alpha,\varepsilon}(t,x,\xi)\phi(t,x)$, and sum the two equations; using the fact that $-f_+^{\alpha,\varepsilon}$ and $-g_-^{\beta,\nu}$ are non-decreasing with respect to ξ, we find, after some integrate by parts,

$$-\int_{\mathbb{R}^{d+2}} f_+^{\alpha,\varepsilon}g_-^{\beta,\nu}(\partial_t + a\cdot\nabla)\phi + \eta\int_{Q\times\mathbb{R}_\xi}\delta_v\nabla v\cdot(\nabla(f_+^{\alpha,\varepsilon}\phi))\star(\check{\gamma_\nu}\otimes\check{\theta_\beta})$$
$$-\int_{\mathbb{R}^{d+2}}\left[f_+^{0\;\varepsilon}\otimes\theta_\alpha g_-^{\beta,\nu} + f_-^{0\;\nu}\otimes\theta_\beta f_+^{\alpha,\varepsilon}\right]\phi \geqslant 0. \tag{11}$$

Thanks to the decentred regularization, $f_+^{\alpha,\varepsilon}(t,x,\xi)$ is null if $t\leqslant\alpha/2$; hence, for $\beta\leqslant\alpha/2$, $f_-^{0\;\nu}\otimes\theta_\beta f_+^{\alpha,\varepsilon}\equiv 0$. Moreover, the function which associates t with

$$\begin{aligned}\int_{\Omega\times\mathbb{R}_\xi} f_+^{0\;\varepsilon}(x,\xi)g_-(t,x,\xi)\phi(t,x)\,dx\,d\xi &= \int_\Omega\int_\Omega\int_{\mathbb{R}_\xi} f_+^0(y,\xi)g_-(t,x,\xi)\gamma_\varepsilon(x-y)\phi(t,x)\,d\xi\,dy\,dx\\ &= -\int_\Omega\int_\Omega (u_0(y)-v(t,x))^+\gamma_\varepsilon(x-y)\phi(t,x)\,d\xi\,dy\,dx \end{aligned}\tag{12}$$

is continuous (because $v\in\mathsf{C}([0,T_0];L^1(\Omega))$ and we have $g_-(0,\cdot,\cdot)=f_-^0$. Therefore, letting β, ν and α successively tend to zero in (11), we have

$$\int_{Q\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)(\partial_t + a\cdot\nabla)\phi + \eta\int_{Q\times\mathbb{R}_\xi}\delta_v\nabla v\cdot\nabla(f_+^\varepsilon\phi) - \int_{\Omega\times\mathbb{R}_\xi} f_+^{0\;\varepsilon}f_-^0\phi^{(t=0)} \geqslant 0.$$

Choose $T\in[0,T_0]$ and let $\phi(t,x)=\lambda(x)w_\beta(t)$ where $w_\beta(t)=\int_{t-T}^{+\infty}\theta_\beta(r)dr$; we obtain

$$\int_{Q\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)[-\theta_\beta(t-T)\lambda + w_\beta a\cdot\nabla\lambda] + \eta\int_{Q\times\mathbb{R}_\xi}\delta_v\nabla v\cdot\nabla(f_+^\varepsilon\lambda)w_\beta - \int_{\Omega\times\mathbb{R}_\xi} f_+^{0\;\varepsilon}f_-^0\lambda \geqslant 0.$$

The function $t\to\int_{\Omega\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)(t,x,\xi)\lambda(x)\,dx$ is continuous (it is similar to (12)); thus, letting $\beta\to 0$,

$$-\int_{\Omega\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)^{(t=T)}\lambda + \int_{Q^T\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)a\cdot\nabla\lambda + \int_{Q^T\times\mathbb{R}_\xi}\eta\delta_v\nabla v\cdot\nabla(f_+^\varepsilon\lambda) - \int_{\Omega\times\mathbb{R}_\xi} f_+^{0\;\varepsilon}f_-^0\lambda \geqslant 0$$

where $Q^T=(0,T)\times\Omega$. We therefore obtain

$$T_1\leqslant T_2+T_{IC}+T_D \tag{13}$$

where

$$\begin{aligned} T_1 &= \int_{\Omega\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)^{(t=T)}\lambda\,,\\ T_2 &= \int_{Q^T\times\mathbb{R}_\xi}(-f_+^\varepsilon g_-)a\cdot\nabla\lambda\,,\\ T_D &= \int_{Q^T\times\mathbb{R}_\xi}\eta\delta_v\nabla v\cdot\nabla(f_+^\varepsilon\lambda)\,,\\ T_{IC} &= -\int_{\Omega\times\mathbb{R}_\xi} f_+^{0\;\varepsilon}f_-^0\lambda. \end{aligned}$$

We now estimate these terms. We have

$$\begin{aligned}
T_1 &= \int_K \int_\Omega \left[\int_{\mathbb{R}_\xi} (-f_+(T,y,\xi) g_-(T,x,\xi))\, d\xi \right] \lambda(x) \gamma_\varepsilon(x-y)\, dy\, dx \\
&= \int_K \int_\Omega (u(T,y) - v(T,x))^+ \lambda(x) \gamma_\varepsilon(x-y)\, dy\, dx \\
&\geqslant \int_K \int_\Omega (u(T,x) - v(T,x))^+ \lambda(x) \gamma_\varepsilon(x-y)\, dy\, dx - \int_K \int_\Omega (u(T,x) - u(T,y))^+ \lambda(x) \gamma_\varepsilon(x-y)\, dy\, dx.
\end{aligned}$$

But, if $x \in K$, we have, by choice of ε, $x - \Omega \supset \mathrm{supp}(\gamma_\varepsilon)$, hence $\int_\Omega \gamma_\varepsilon(x-y)\, dy = 1$. Moreover, since $u(T,\cdot) \in \mathsf{BV}(\Omega)$, by Lemma 2 (see the appendix),

$$\int_K \int_\Omega (u(T,x) - u(T,y))^+ \lambda(x) \gamma_\varepsilon(x-y)\, dy\, dx \leqslant \int_\Omega \int_\Omega |u(T,x) - u(T,y)| \gamma_\varepsilon(x-y)\, dy\, dx \leqslant C\varepsilon.$$

Hence,

$$T_1 \geqslant \int_\Omega (u(T,x) - v(T,x))^+ \lambda(x)\, dx - C\varepsilon.$$

Next, reasoning as for T_1,

$$\begin{aligned}
T_2 &= \int_0^T \int_K \int_\Omega \int_{\mathbb{R}_\xi} (-f_+(t,y,\xi) g_-(t,x,\xi)) \gamma_\varepsilon(x-y) a(\xi) \cdot \nabla \lambda(x)\, dy\, dx\, dt \\
&\leqslant C \int_0^T \int_\Omega \int_\Omega (u(t,y) - v(t,x))^+ \gamma_\varepsilon(x-y)\, dy\, dx\, dt \\
&\leqslant C\varepsilon + C \int_0^T \int_\Omega (u(t,x) - v(t,x))^+\, dx\, dt.
\end{aligned}$$

Let us estimate the diffusion term T_D. First, we write: $T_D = T_D^1 + T_D^2$ with

$$T_D^1 = \int_{Q^T \times \mathbb{R}_\xi} \eta \delta_v \nabla v \cdot f_+^\varepsilon \nabla \lambda = \int_{Q^T \times \mathbb{R}_\xi} \eta \nabla v \cdot \left(\int_{\mathbb{R}_\xi} \delta_v f_+^\varepsilon \right) \nabla \lambda \leqslant \eta \int_{Q^T} |\nabla v| |\nabla \lambda| \leqslant C\eta$$

and

$$T_D^2 = \int_{Q^T \times \mathbb{R}_\xi} \eta \delta_v \nabla v \cdot \lambda \nabla f_+^\varepsilon \leqslant C\eta \int_{Q^T} |\nabla v|(t,x) \sup_\xi |\nabla f_+^\varepsilon|(t,x,\xi)\, dt dx.$$

But $\nabla f_+^\varepsilon(t,x,\xi) = \int_\Omega f_+(t,y,\xi) \nabla \gamma_\varepsilon(x-y)\, dy$, so that $|\nabla f_+^\varepsilon|(t,x,\xi) \leqslant ||\nabla \gamma_\varepsilon||_{L^1(\mathbb{R}^d)} \leqslant C/\varepsilon$. Hence,

$$T_D^2 \leqslant \frac{C\eta}{\varepsilon} \int_{Q^T} |\nabla v| \leqslant \frac{C\eta}{\varepsilon}.$$

Using Lemma 2, a straightforward computation gives $T_{IC} \leqslant C\varepsilon$. We finally gather the different estimates in (13) and get, for all ε,

$$\int_\Omega (u(T,x) - v(T,x))^+ \lambda(x) dx \leqslant C \left(\varepsilon + \frac{\eta}{\varepsilon} \right) + C \int_0^T \int_\Omega (u(t,x) - v(t,x))^+\, dx\, dt.$$

Minimizing on ε, we obtain (recall that $\lambda = \lambda_0$ here)

$$\int_\Omega (u(T,x) - v(T,x))^+ \lambda_0(x) dx \leqslant C\sqrt{\eta} + C \int_0^T \int_\Omega (u(t,x) - v(t,x))^+\, dx\, dt. \tag{14}$$

4 Transport and regularization of the kinetic equations

In order to estimate $(u(T,\cdot)-v(T,\cdot))^+$ near the boundary of Ω, we choose a chart (O_i,h_i,λ_i) and we transport the equations to B^d_+. In the following, we drop the subscript i.

4.1 Transport of the kinetic equations

We now write the kinetic equations satisfied by u and v once they have been transported on B^d_+. Consider a test function $\Psi\in\mathsf{C}^\infty_\mathsf{c}(\mathbb{R}\times B^d\times\mathbb{R}_\xi)$ and set $\phi(t,x,\xi)=\Psi(t,h(x),\xi)\in\mathsf{C}^2_\mathsf{c}(\mathbb{R}_t\times O\times\mathbb{R}_\xi)$. Next, extend ϕ by 0 to get a function $\phi\in\mathsf{C}^2_\mathsf{c}(\mathbb{R}^{d+2})$ and plug it into $(5)_+$ (ϕ is not C^∞ but is regular enough to be taken as a test function in this equation). This gives

$$\int_{\mathbb{R}\times O\times\mathbb{R}_\xi} f_+\Big[(\partial_t\Psi)\circ h+a\cdot {h'}^T(\nabla\Psi)\circ h\Big]+\int_{O\times\mathbb{R}_\xi} f^0_+\Psi^{(t=0)}\circ h+\int_{\mathbb{R}\times(\partial\Omega\cap O)\times\mathbb{R}_\xi}(-a\cdot n)f^\tau_+\overline{\Psi}\circ h \\ =\int_{\mathbb{R}\times O\times\mathbb{R}_\xi}(\partial_\xi\Psi)\circ h\,dm.$$

Through the change of variables $y=h(x)$, and by definition of the measure on Σ, we obtain

$$\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}|Jh^{-1}|f_+\circ h^{-1}(\partial_t\Psi+h'\circ h^{-1}a\cdot\nabla\Psi)+\int_{B^d_+}\int_{\mathbb{R}_\xi}|Jh^{-1}|f^0_+\circ h^{-1}\Psi^{(t=0)} \\ +\int_0^\infty\int_{B^{d-1}}\int_{\mathbb{R}_\xi}(-a\cdot nf^\tau_+)\circ h^{-1}\overline{\Psi}\left|\frac{\partial h^{-1}}{\partial x_1}\wedge\cdots\wedge\frac{\partial h^{-1}}{\partial x_{d-1}}\right|dx_1\dots dx_{d-1} \\ =\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}(\partial_\xi\Psi)\,d(h_*m).$$

In the following, we adopt the notations

$$j(x)=|Jh^{-1}(x)|\quad\text{ and }\quad H(x)=h'\circ h^{-1}(x)\quad\text{ and }\quad l(x)=\left|\frac{\partial h^{-1}}{\partial x_1}\wedge\cdots\wedge\frac{\partial h^{-1}}{\partial x_{d-1}}\right|(x).$$

Moreover, for any function $r(t,x,\xi)$, we write $\tilde r(t,x,\xi)$ for $r(t,h^{-1}(x),\xi)$. Therefore, the previous equality reads

$$\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}j\tilde{f_+}(\partial_t\Psi+Ha\cdot\nabla\Psi)+\int_{B^d_+}\int_{\mathbb{R}_\xi}j\tilde{f^0_+}\Psi^{(t=0)}+\int_0^\infty\int_{B^{d-1}}\int_{\mathbb{R}_\xi}l(-a\cdot\tilde n)\tilde{f^\tau_+}\overline{\Psi} \\ =\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}\partial_\xi\Psi\,d(h_*m).\quad(15)$$

Similar computations are achieved on the kinetic equation satisfied by v. We obtain

$$\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}j\tilde{g_-}(\partial_t\Psi+Ha\cdot\nabla\Psi)+\int_{B^d_+}\int_{\mathbb{R}_\xi}j\tilde{f^0_-}\Psi^{(t=0)} \\ +\int_0^\infty\int_{B^{d-1}}\int_{\mathbb{R}_\xi}l(-a\cdot\tilde n)\tilde{f^b_-}\overline{\Psi}+\int_0^\infty\int_{B^{d-1}}\int_{\mathbb{R}_\xi}l\tilde D\delta_{\tilde{u_b}}\overline{\Psi} \\ +\eta\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}\tilde Z\delta_{\tilde v}\cdot\nabla\Psi=\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}\partial_\xi\Psi\,d(h_*q)\quad(16)$$

where $D(t,\overline{x})=\eta\nabla v(t,\overline{x})\cdot n(\overline{x})$ and $Z(t,x)=-h'(x)\nabla v(t,x)$. Notice that

$$Z\text{ is bounded in }L^1((0,T)\times\Omega)\text{ for all }T\geqslant 0.\quad(17)$$

This property, as well as the Lipschitz continuity $[0,\infty)\to\mathsf{L}^1$ of u and v (with a Lipschitz constant for v independent of η) and the bounds on $|u(t,\cdot)|_{\mathsf{BV}}$ and $|v(t,\cdot)|_{\mathsf{BV}}$, are conserved by the transport by h.

4.2 Transport of the BLN condition

We state here the only consequence of (6) that we use in the following.

Let $\Psi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R} \times B^{d-1} \times \mathbb{R}_\xi)$ be non-negative and non-decreasing with respect to ξ. The function $\phi(t, \overline{x}, \xi) = \Psi(t, h(\overline{x}), \xi)(1 - f^b_+(t, \overline{x}, \xi))$ is non-decreasing with respect to ξ (since Ψ and $1 - f^b_+$ are non-negative and non-decreasing with respect to ξ). Hence, (6) implies

$$\int_0^\infty \int_{\partial\Omega\cap O} \int_{\mathbb{R}_\xi} (-a \cdot n) f^\tau_+ \Psi \circ h(1 - f^b_+) \leqslant \int_0^\infty \int_{\partial\Omega\cap O} \int_{\mathbb{R}_\xi} M f^b_+(1 - f^b_+) \Psi \circ h.$$

But $f^b_+(1 - f^b_+) = 0$ so that, transporting this equation with h^{-1} on B^{d-1}, we deduce that, for all $\Psi \in \mathsf{C}^\infty_\mathsf{c}(\mathbb{R} \times B^{d-1} \times \mathbb{R}_\xi)$ which is non-negative and non-decreasing with respect to ξ,

$$\int_0^\infty \int_{B^{d-1}} \int_{\mathbb{R}_\xi} l(-a \cdot \tilde{n}) \tilde{f^\tau_+} \Psi \leqslant \int_0^\infty \int_{B^{d-1}} \int_{\mathbb{R}_\xi} l(-a \cdot \tilde{n}) \tilde{f^\tau_+} \tilde{f^b_+} \Psi. \tag{18}$$

We also need to understand how the unit normal is transported by the chart (O, h).

Lemma 1 *For all $\overline{y} \in B^{d-1}$ and all $X \in \mathbb{R}^d$, we have $l(\overline{y}) X \cdot \tilde{n}(\overline{y}) = -j(\overline{y})(H(\overline{y})X)_d$, where $(H(\overline{y})X)_d$ is the d-th coordinate of $H(\overline{y})X$.*

Proof of Lemma 1
Let $\psi \in \mathsf{C}^\infty_\mathsf{c}(B^d)$ and $\phi = \psi \circ h \in \mathsf{C}^2_\mathsf{c}(O)$ (extented by 0 outside O). Integrating by parts, we have

$$\int_\Omega X \cdot \nabla\phi(x)\, dx = \int_{\partial\Omega} \phi(\overline{x}) X \cdot n(\overline{x})\, d\sigma(\overline{x}).$$

Since $\nabla\phi(x) = h'(x)^T \nabla\psi(h(x))$, transporting these integrals by h (all the integrands are null outside O), we find

$$\int_{B^d_+} j(x) H(x) X \cdot \nabla\psi(x)\, dx = \int_{B^d_+} X \cdot (h'(h^{-1}(x)))^T \nabla\psi(x) |Jh^{-1}(x)|\, dx = \int_{B^{d-1}} \psi(\overline{x}) X \cdot n(h^{-1}(\overline{x})) l(\overline{x})\, d\overline{x}.$$

Another integrate by parts then yields

$$\int_{B^{d-1}} \psi(\overline{x}) X \cdot n(h^{-1}(\overline{x})) l(\overline{x})\, d\overline{x} = \int_{B^{d-1}} (-j(\overline{x})(H(\overline{x})X)_d) \psi(\overline{x})\, d\overline{x} - \int_{B^d_+} \mathrm{div}(jHX)(x) \psi(x)\, dx.$$

(the unit normal to B^d_+ on B^{d-1} is $(0, \dots, 0, -1)$). Taking first $\psi \in \mathsf{C}^\infty_\mathsf{c}(B^d_+)$, we see that $\mathrm{div}(jHX) = 0$ on B^d_+; thus, for all $\psi \in \mathsf{C}^\infty_\mathsf{c}(B^d)$, $\int_{B^{d-1}} \psi(\overline{x}) X \cdot n(h^{-1}(\overline{x})) l(\overline{x})\, d\overline{x} = \int_{B^{d-1}} (-j(\overline{x})(H(\overline{x})X)_d) \psi(\overline{x})\, d\overline{x}$, which concludes the proof. ■

4.3 Regularization of the transported equations

From now on, we work on B^d and we thus simply write r for $\tilde{r}$. Let $K := \mathrm{supp}(\lambda)$ (compact subset of B^d). We now regularize equations (15) and (16).

For $\overline{\varepsilon} > 0$, we denote $\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}) = \prod_{i=1}^{d-1} \theta_{\overline{\varepsilon}}(x_i)$; we take $\varepsilon_d > 0$ and we denote $\varepsilon = (\overline{\varepsilon}, \varepsilon_d)$, $\gamma_\varepsilon(x) = \overline{\gamma}_{\overline{\varepsilon}}(\overline{x}) \theta_{\varepsilon_d}(x_d)$. We choose $\overline{\varepsilon} + \varepsilon_d < \mathrm{dist}(K, \partial B^d)$. Let $\Psi \in \mathsf{C}^2_\mathsf{c}(\mathbb{R} \times B^d \times \mathbb{R}_\xi)$ with support in $\mathbb{R} \times K \times \mathbb{R}_\xi$; then, $\Psi \star (\check{\gamma}_\varepsilon \otimes \check{\theta}_\alpha)$ is compactly supported in $\mathbb{R} \times B^d \times \mathbb{R}_\xi$. Using $\Psi \star (\check{\gamma}_\varepsilon \otimes \check{\theta}_\alpha)$ in (15), we get

$$\int_{\mathbb{R}^{d+2}} (jf_+)^{\alpha,\varepsilon} \partial_t \Psi + (jf_+ H)^{\alpha,\varepsilon} a \cdot \nabla\Psi + \int_{\mathbb{R}^{d+2}} (jf^0_+)^\varepsilon \otimes \theta_\alpha \Psi \\ + \int_{\mathbb{R}^{d+2}} (l(-a \cdot n) f^\tau_+)^{\alpha,\overline{\varepsilon}} \otimes \theta_{\varepsilon_d} \Psi = \int_{\mathbb{R}^{d+2}} \partial_\xi \Psi\, d(h_* m)^{\alpha,\varepsilon}. \tag{19}$$

The same test function with parameters β and ν in (16) gives

$$\int_{\mathbb{R}^{d+2}}(jg_-)^{\beta,\nu}\partial_t\Psi+(jg_-H)^{\beta,\nu}a\cdot\nabla\Psi+\int_{\mathbb{R}^{d+2}}(jf_-^0)^{\nu}\otimes\theta_\beta\Psi$$
$$+\int_{\mathbb{R}^{d+2}}(l(-a\cdot n)f_-^b)^{\beta,\overline{\nu}}\otimes\theta_{\nu_d}\Psi+\int_0^\infty\int_{B^{d-1}}\int_{\mathbb{R}_\xi}lD\delta_{u_b}\overline{\Psi\star(\check{\gamma_\nu}\otimes\check{\theta_\beta})}$$
$$+\eta\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}Z\delta_v\cdot\nabla\Psi\star(\check{\gamma_\nu}\otimes\check{\theta_\beta})=\int_{\mathbb{R}^{d+2}}\partial_\xi\Psi\,d(h_*q)^{\beta,\nu}. \quad (20)$$

5 Combination of the equations and new estimates

The next step consists in combining the two preceding kinetic equations. Choose a non-negative regular function $\phi(t,x)$, with support in $]-\infty,T_0]\times K$, and apply $(-jf_+)^{\alpha,\varepsilon}(t,x,\xi)\phi(t,x)$ as a test function in (20) and $(-jg_-)^{\beta,\nu}(t,x,\xi)\phi(t,x)$ as a test function in (19). These two test functions are non-decreasing with respect to ξ so that, summing the results, we get $U_1^{\beta,\nu}+U_2^{\beta,\nu}+U_3^{\beta,\nu}+U_4^{\beta,\nu}+U_5^{\beta,\nu}+U_6^{\beta,\nu}\geqslant 0$, where

$$\begin{aligned}
U_1^{\beta,\nu} &= \int_{\mathbb{R}^{d+2}}(jf_+)^{\alpha,\varepsilon}(\partial_t(-jg_-)^{\beta,\nu}\phi+(-jg_-)^{\beta,\nu}\partial_t\phi)+(-jg_-)^{\beta,\nu}(\partial_t(jf_+)^{\alpha,\varepsilon}\phi+(jf_+)^{\alpha,\varepsilon}\partial_t\phi)\\
U_2^{\beta,\nu} &= \int_{\mathbb{R}^{d+2}}(jf_+H)^{\alpha,\varepsilon}a\cdot\big(\nabla(-jg_-)^{\beta,\nu}\phi+(-jg_-)^{\beta,\nu}\nabla\phi\big)\\
&\qquad +(-jg_-H)^{\beta,\nu}a\cdot(\nabla(jf_+)^{\alpha,\varepsilon}\phi+(jf_+)^{\alpha,\varepsilon}\nabla\phi)\\
U_3^{\beta,\nu} &= \int_{\mathbb{R}^{d+2}}(jf_+^0)^{\varepsilon}\otimes\theta_\alpha(-jg_-)^{\beta,\nu}\phi+(-jf_-^0)^{\nu}\otimes\theta_\beta(jf_+)^{\alpha,\varepsilon}\phi\\
U_4^{\beta,\nu} &= -\int_0^\infty\int_{B^{d-1}}\int_{\mathbb{R}_\xi}lD\delta_{u_b}\overline{((jf_+)^{\alpha,\varepsilon}\phi)\star(\check{\gamma_\nu}\otimes\check{\theta_\beta})}\\
U_5^{\beta,\nu} &= \int_{\mathbb{R}^{d+2}}(l(-a\cdot n)f_+^\tau)^{\alpha,\overline{\varepsilon}}\otimes\theta_{\varepsilon_d}(-jg_-)^{\beta,\nu}\phi+(-l(-a\cdot n)f_-^b)^{\beta,\overline{\nu}}\otimes\theta_{\nu_d}(jf_+)^{\alpha,\varepsilon}\phi\\
U_6^{\beta,\nu} &= \eta\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}\delta_vZ\cdot\nabla((jf_+)^{\alpha,\varepsilon}\phi)\star(\check{\gamma_\nu}\otimes\check{\theta_\beta}).
\end{aligned}$$

5.1 Passing to the limit in β and ν

We study the limits of $U_1^{\beta,\nu},\dots,U_6^{\beta,\nu}$ as β and ν tend to 0.

The first term $U_1^{\beta,\nu}$
Integrating by parts, we have

$$U_1^{\beta,\nu}=\int_{\mathbb{R}^{d+2}}(jf_+)^{\alpha,\varepsilon}(-jg_-)^{\beta,\nu}\partial_t\phi$$

and thus, as $\beta\to 0$ and $\nu\to 0$,

$$U_1^{\beta,\nu}\to\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}(-jg_-)\partial_t\phi. \quad (21)$$

The second term $U_2^{\beta,\nu}$
The first step, here, is to get H out of the regularizations $(jf_+H)^{\alpha,\varepsilon}$ and $(jg_-H)^{\beta,\nu}$. To do this, we

notice that, for all $(t,x,\xi)\in\mathbb{R}\times B^d\times\mathbb{R}$, since H is C^1,

$$\begin{aligned}
|(jf_+H)^{\alpha,\varepsilon}(t,x,\xi)-&H(x)(jf_+)^{\alpha,\varepsilon}(t,x,\xi)|\\
&= \left|\int_0^\infty\int_{B^d_+} j(y)f_+(s,y,\xi)H(y)\theta_\alpha(t-s)\gamma_\varepsilon(x-y)\,ds\,dy\right.\\
&\qquad\left.-H(x)\int_0^\infty\int_{B^d_+} j(y)f_+(s,y,\xi)\theta_\alpha(t-s)\gamma_\varepsilon(x-y)\,ds\,dy\right|\\
&\leqslant \int_0^\infty\int_{B^d_+} j(y)f_+(s,y,\xi)|H(y)-H(x)|\theta_\alpha(t-s)\gamma_\varepsilon(x-y)\,ds\,dy\\
&\leqslant C(\overline{\varepsilon}+\varepsilon_d)\int_0^\infty\int_{B^d_+}\theta_\alpha(t-s)\gamma_\varepsilon(x-y)\,ds\,dy\leqslant C(\overline{\varepsilon}+\varepsilon_d)
\end{aligned}$$

(here, "$|\cdot|$" is a matrice norm). Hence,

$$\begin{aligned}
&\left|\int_{\mathbb{R}\times B^d\times\mathbb{R}_\xi}(jf_+H)^{\alpha,\varepsilon}a\cdot\big(\nabla(-jg_-)^{\beta,\nu}\phi+(-jg_-)^{\beta,\nu}\nabla\phi\big)\right.\\
&\qquad\left.-\int_{\mathbb{R}\times B^d\times\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}Ha\cdot\big(\nabla(-jg_-)^{\beta,\nu}\phi+(-jg_-)^{\beta,\nu}\nabla\phi\big)\right|\\
&\leqslant C(\overline{\varepsilon}+\varepsilon_d)\left(||\nabla(-jg_-)^{\beta,\nu}||_{\mathsf{L}^1(]-\infty,T_0]\times K\times[-D,D])}||\phi||_\infty+||(-jg_-)^{\beta,\nu}||_{\mathsf{L}^\infty(\mathbb{R}^{d+2})}||\nabla\phi||_\infty\right)
\end{aligned}\tag{22}$$

(recall that $\mathrm{supp}(\phi)\subset]-\infty,T_0]\times K$). But, by Lemma 3 (see the appendix), for all $s\in\mathbb{R}^+$,

$$\int_K\int_{-D}^D|\nabla(j\mathrm{sgn}_-(v(s,\cdot)-\xi))^\nu|\leqslant C(1+|v(s,\cdot)|_{\mathsf{BV}(B^d_+)})\leqslant C.$$

Therefore,

$$\begin{aligned}
||\nabla(-jg_-)^{\beta,\nu}||_{\mathsf{L}^1(]-\infty,T_0]\times K\times[-D,D])} &\leqslant \int_{-\infty}^{T_0}\int_0^\infty\int_K\int_{-D}^D|\nabla(j\mathrm{sgn}_-(v(s,\cdot)-\xi))^\nu|\theta_\beta(t-s)\,ds\,dt\\
&\leqslant C\int_0^{T_0}\int_0^\infty\theta_\beta(t-s)\,ds\,dt\leqslant C.
\end{aligned}\tag{23}$$

Noticing that, thanks to ϕ, the integrals in $U_2^{\beta,\nu}$ are in fact on $\mathbb{R}\times B^d\times\mathbb{R}_\xi$, we deduce from (22), (23) and similar estimates for the second part of $U_2^{\beta,\nu}$ that $U_2^{\beta,\nu}$ is equal to

$$\begin{aligned}
&\int_{\mathbb{R}\times B^d\times\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}Ha\cdot(\nabla(-jg_-)^{\beta,\nu}\phi+(-jg_-)^{\beta,\nu}\nabla\phi)+(-jg_-)^{\beta,\nu}Ha\cdot(\nabla(jf_+)^{\alpha,\varepsilon}\phi+(jf_+)^{\alpha,\varepsilon}\nabla\phi)\\
&\qquad+\mathcal{O}((\overline{\varepsilon}+\varepsilon_d+\overline{\nu}+\nu_d)(||\phi||_\infty+||\nabla\phi||_\infty))\\
&=\int_{\mathbb{R}\times B^d\times\mathbb{R}_\xi}\phi Ha\cdot\nabla((jf_+)^{\alpha,\varepsilon}(-jg_-)^{\beta,\nu})+2(jf_+)^{\alpha,\varepsilon}(-jg_-)^{\beta,\nu}Ha\cdot\nabla\phi\\
&\qquad+\mathcal{O}((\overline{\varepsilon}+\varepsilon_d+\overline{\nu}+\nu_d)(||\phi||_\infty+||\nabla\phi||_\infty))\\
&=\int_{\mathbb{R}\times B^d\times\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}(-jg_-)^{\beta,\nu}(2Ha\cdot\nabla\phi-\mathrm{div}(\phi Ha))+\mathcal{O}((\overline{\varepsilon}+\varepsilon_d+\overline{\nu}+\nu_d)(||\phi||_\infty+||\nabla\phi||_\infty))
\end{aligned}$$

(we used the fact that ϕ has a compact support in $\mathbb{R}\times B^d$). Letting β and ν tend to 0, this gives

$$\limsup_{\beta,\nu\to0}U_2^{\beta,\nu}\leqslant\mathcal{O}((\overline{\varepsilon}+\varepsilon_d)(||\phi||_\infty+||\nabla\phi||_\infty))+\int_0^\infty\int_{B^d_+}\int_{\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}(-jg_-)(2Ha\cdot\nabla\phi-\mathrm{div}(\phi Ha)).\tag{24}$$

The third, fourth and fifth terms
By the choice of a decentred convolution kernel, we have for β and ν_d small enough

$$\theta_\beta(\cdot)(jf_+)^{\alpha,\varepsilon}(\cdot,x,\xi)\equiv 0, \qquad \overline{((jf_+)^{\alpha,\varepsilon}\phi)\star(\check{\gamma}_\nu\otimes\check{\theta}_\beta)}\equiv 0, \qquad \theta_{\nu_d}(\cdot)(jf_+)^{\alpha,\varepsilon}(t,\overline{x},\cdot,\xi)\equiv 0.$$

Therefore, as β and ν go to 0,

$$U_3^{\beta,\nu} \rightarrow \int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(jf_+^0)^\varepsilon\otimes\theta_\alpha(-jg_-)\phi \tag{25}$$

$$U_4^{\beta,\nu} \rightarrow 0 \tag{26}$$

$$U_5^{\beta,\nu} \rightarrow \int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(l(-a\cdot n)f_+^\tau)^{\alpha,\overline{\varepsilon}}\otimes\theta_{\varepsilon_d}(-jg_-)\phi. \tag{27}$$

The sixth term $U_6^{\beta,\nu}$
Since $(jf_+)^{\alpha,\varepsilon}\phi$ is regular, we have

$$U_6^{\beta,\nu}\rightarrow\eta\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}\delta_v Z\cdot\nabla((jf_+)^{\alpha,\varepsilon}\phi). \tag{28}$$

as β and ν tend to 0.

Using (21), (24), (25), (26), (27) and (28) in $U_1^{\beta,\nu}+U_2^{\beta,\nu}+U_3^{\beta,\nu}+U_4^{\beta,\nu}+U_5^{\beta,\nu}+U_6^{\beta,\nu}\geqslant 0$, we obtain as β and ν go to 0

$$\begin{aligned}
&-\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}(-jg_-)\partial_t\phi \\
&\leqslant\ C((\overline{\varepsilon}+\varepsilon_d)(||\phi||_\infty+||\nabla\phi||_\infty))+\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}(-jg_-)(2Ha\cdot\nabla\phi-\mathrm{div}(\phi Ha)) \\
&\quad+\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(jf_+^0)^\varepsilon\otimes\theta_\alpha(-jg_-)\phi+\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}(l(-a\cdot n)f_+^\tau)^{\alpha,\overline{\varepsilon}}\otimes\theta_{\varepsilon_d}(-jg_-)\phi \\
&\quad+\eta\int_0^\infty\int_{B_+^d}\int_{\mathbb{R}_\xi}\delta_v Z\cdot\nabla((jf_+)^{\alpha,\varepsilon}\phi).
\end{aligned} \tag{29}$$

5.2 Choice of ϕ and continuation of the estimates

We now take $T\in[0,T_0]$ and $\phi(t,x)=\lambda(x)w_\beta(t)$, where $w_\beta(t)=\int_{t-T}^\infty\check{\theta}_\beta(s)\,ds$ (notice that w_β has its support in $]-\infty,T_0]$). The function w_β converges, as $\beta\to 0$, to the characteristic function of $]-\infty,T]$ and $w'_\beta(t)=-\check{\theta}_\beta(t-T)$ converges to $-\delta_T$. Since $t\to\int_{B_+^d}\int_{\mathbb{R}_\xi}(jf_+)^{\alpha,\varepsilon}(t,x,\xi)(-jg_-)(t,x,\xi)\lambda(x)\,dx\,d\xi$ is continuous (it is similar to (12)), we deduce from (29) that

$$T_1^{\alpha,\varepsilon}\leq T_2^{\alpha,\varepsilon}+T_3^{\alpha,\varepsilon}+T_4^{\alpha,\varepsilon}+T_5^{\alpha,\varepsilon}+C(\overline{\varepsilon}+\varepsilon_d) \tag{30}$$

where

$$
\begin{aligned}
T_1^{\alpha,\varepsilon} &= \int_{B_+^d}\int_{\mathbb{R}_\xi} ((jf_+)^{\alpha,\varepsilon}(-jg_-))^{(t=T)}\lambda \\
T_2^{\alpha,\varepsilon} &= \int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} (jf_+)^{\alpha,\varepsilon}(-jg_-)Y \\
T_3^{\alpha,\varepsilon} &= \int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} (jf_+^0)^{\varepsilon}\otimes\theta_\alpha(-jg_-)\lambda \\
T_4^{\alpha,\varepsilon} &= \eta\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} \delta_v Z\cdot\nabla((jf_+)^{\alpha,\varepsilon}\lambda) \\
T_5^{\alpha,\varepsilon} &= \int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} (l(-a\cdot n)f_+^\tau)^{\alpha,\overline{\varepsilon}}\otimes\theta_{\varepsilon_d}(-jg_-)\lambda
\end{aligned}
$$

and $Y = 2Ha\cdot\nabla\lambda - \operatorname{div}(\lambda Ha) \in \mathsf{L}^\infty((0,T_0)\times B_+^d\times\mathbb{R}_\xi)$. Our aim is to obtain an inequality of the kind of (14); we now estimate each term $T_i^{\alpha,\varepsilon}$.

The first term $T_1^{\alpha,\varepsilon}$

We have

$$
\begin{aligned}
T_1^{\alpha,\varepsilon} &= \int_{B_+^d}\int_0^\infty\int_{B_+^d} j(y)j(x)(u(s,y)-v(T,x))^+\theta_\alpha(T-s)\gamma_\varepsilon(x-y)\lambda(x)\,dy\,ds\,dx \\
&\geqslant \int_{B_+^d}\int_0^\infty\int_{B_+^d} j(y)j(x)(u(T,x)-v(T,x))^+\theta_\alpha(T-s)\gamma_\varepsilon(x-y)\lambda(x)\,dy\,ds\,dx \\
&\quad -\int_{B_+^d}\int_0^\infty\int_{B_+^d} j(y)j(x)(u(T,x)-u(s,y))^+\theta_\alpha(T-s)\gamma_\varepsilon(x-y)\lambda(x)\,dy\,ds\,dx.
\end{aligned}
\tag{31}
$$

Lemma 2 and Proposition 1 give

$$
\int_{B_+^d}\int_0^\infty\int_{B_+^d} |u(T,x)-u(s,y)|\theta_\alpha(T-s)\gamma_\varepsilon(x-y)\,dy\,ds\,dx \leqslant C(\overline{\varepsilon}+\varepsilon_d+\alpha). \tag{32}
$$

Since j is bounded from below by $\underline{j}>0$, we have

$$
\begin{aligned}
&\int_{B_+^d}\int_0^\infty\int_{B_+^d} j(y)j(x)(u(T,x)-v(T,x))^+\theta_\alpha(T-s)\gamma_\varepsilon(x-y)\lambda(x)\,dy\,ds\,dx \\
&\geqslant \underline{j}^2\int_{B_+^d}\lambda(x)(u(T,x)-v(T,x))^+\left(\int_0^\infty\theta_\alpha(T-s)\,ds\right)\left(\int_{B_+^d}\gamma_\varepsilon(x-y)\,dy\right)dx \\
&\geqslant \underline{j}^2\int_{K\cap B_+^d\cap\{x_d>\varepsilon_d\}}\lambda(x)(u(T,x)-v(T,x))^+\left(\int_0^\infty\theta_\alpha(T-s)\,ds\right)\left(\int_{B_+^d}\gamma_\varepsilon(x-y)\,dy\right)dx
\end{aligned}
$$

(recall that K is the support of λ).

If $T\geqslant\alpha$, then $\int_0^\infty\theta_\alpha(T-s)\,ds = \int_{-\infty}^T\theta_\alpha = 1$. Moreover, if $x\in K$ and $x_d>\varepsilon_d$, we have $]0,\overline{\varepsilon}[^{d-1}\times]0,\varepsilon_d[\subset x-B_+^d$ (indeed, if $z\in]0,\overline{\varepsilon}[^{d-1}\times]0,\varepsilon_d[$ then, since $x\in K$, we have $x-z\in B^d$ and, since $x_d>\varepsilon_d>z_d$, $x-z\in B_+^d$); hence, for those x's, $\int_{B_+^d}\gamma_\varepsilon(x-y)\,dy=1$ since the support of γ_ε is contained in $]0,\overline{\varepsilon}[^{d-1}\times]0,\varepsilon_d[$.

Thus, for $T \geqslant \alpha$,

$$\underline{j}^2 \int_{B_+^d \cap \{x_d > \varepsilon_d\}} \lambda(x)(u(T,x) - v(T,x))^+ \, dx$$
$$\leqslant \int_{B_+^d} \int_0^\infty \int_{B_+^d} j(y) j(x) (u(T,x) - v(T,x))^+ \theta_\alpha(T-s) \gamma_\varepsilon(x-y) \lambda(x) \, dy \, ds \, dx.$$

Since u and v are bounded,

$$\int_{B_+^d \cap \{x_d \leqslant \varepsilon_d\}} \lambda(x)(u(T,x) - v(T,x))^+ \, dx \leqslant \int_{B_+^d \cap \{x_d \leqslant \varepsilon_d\}} C \, dx \leqslant C\varepsilon_d.$$

Hence, if $T \geqslant \alpha$,

$$\underline{j}^2 \int_{B_+^d} \lambda(x)(u(T,x) - v(T,x))^+ \, dx$$
$$\leqslant \int_{B_+^d} \int_0^\infty \int_{B_+^d} j(y) j(x) (u(T,x) - v(T,x))^+ \theta_\alpha(T-s) \gamma_\varepsilon(x-y) \lambda(x) \, dy \, ds \, dx + C\varepsilon_d. \quad (33)$$

Equations (31), (32) and (33) give, if $T \geqslant \alpha$,

$$T_1^{\alpha,\varepsilon} \geqslant -C(\overline{\varepsilon} + \varepsilon_d + \alpha) + \underline{j}^2 \int_{B_+^d} \lambda(x)(u(T,x) - v(T,x))^+ \, dx.$$

But u and v are Lipschitz continuous $[0, T_0] \to \mathsf{L}^1(B_+^d)$ (with a Lipschitz constant not depending on η) and equal to u_0 at $t = 0$; hence, for $T \leqslant \alpha$,

$$\int_{B_+^d} \lambda(x)(u(T,x) - v(T,x))^+ \, dx \leqslant C \int_{B_+^d} |u(T,x) - u_0(x)| \, dx + C \int_{B_+^d} |v(T,x) - u_0(x)| \, dx \leqslant C\alpha.$$

Therefore, $T_1^{\alpha,\varepsilon}$ being non-negative, we have, for all $T \in [0, T_0]$,

$$T_1^{\alpha,\varepsilon} \geqslant -C(\overline{\varepsilon} + \varepsilon_d + \alpha) + \underline{j}^2 \int_{B_+^d} \lambda(x)(u(T,x) - v(T,x))^+ \, dx. \quad (34)$$

The second term $T_2^{\alpha,\varepsilon}$

We have

$$\begin{aligned}
T_2^{\alpha,\varepsilon} &= \int_0^T \int_{B_+^d} \int_0^\infty \int_{B_+^d} \int_{\mathbb{R}_\xi} j(y) j(x) f_+(s,y,\xi)(-g_-(t,x,\xi)) Y(t,x) \theta_\alpha(t-s) \gamma_\varepsilon(x-y) \, d\xi \, dy \, ds \, dx \, dt \\
&\leqslant C \int_0^T \int_{B_+^d} \int_0^\infty \int_{B_+^d} (u(s,y) - v(t,x))^+ \theta_\alpha(t-s) \gamma_\varepsilon(x-y) \, dy \, ds \, dx \, dt \\
&\leqslant C \int_0^T \int_{B_+^d} \int_0^\infty \int_{B_+^d} (u(s,y) - u(t,x))^+ \theta_\alpha(t-s) \gamma_\varepsilon(x-y) \, dy \, ds \, dx \, dt \\
&\quad + C \int_0^T \int_{B_+^d} \int_0^\infty \int_{B_+^d} (u(t,x) - v(t,x))^+ \theta_\alpha(t-s) \gamma_\varepsilon(x-y) \, dy \, ds \, dx \, dt \\
&\leqslant C(\overline{\varepsilon} + \varepsilon_d + \alpha) + C \int_0^T \int_{B_+^d} (u(t,x) - v(t,x))^+ \, dx \, dt
\end{aligned}$$

(we used (32) with $T = t$).

Therefore,

$$T_2^{\alpha,\varepsilon} \leqslant C(\overline{\varepsilon} + \varepsilon_d + \alpha) + C\int_0^T \int_{B_+^d} (u(t,x) - v(t,x))^+ \, dx \, dt. \tag{35}$$

The third term $T_3^{\alpha,\varepsilon}$
We write

$$\begin{aligned} T_3^{\alpha,\varepsilon} &= \int_0^T \int_{B_+^d} \int_{B_+^d} \int_{\mathbb{R}_\xi} j(y)j(x) f_+^0(y,\xi)(-g_-(t,x,\xi))\theta_\alpha(t)\gamma_\varepsilon(x-y)\lambda(x) \, d\xi \, dy \, dx \, dt \\ &\leqslant C\int_0^T \int_{B_+^d} \int_{B_+^d} (u_0(y) - v(t,x))^+ \theta_\alpha(t)\gamma_\varepsilon(x-y) \, dy \, dx \, dt. \end{aligned}$$

But $v(0,x) = u_0(x)$ so that, v being Lipschitz continuous $[0,T_0] \to \mathsf{L}^1(B_+^d)$ (with a Lipschitz constant not depending on η) and u_0 being in $\mathsf{BV}(B_+^d)$, by Lemma 2,

$$\begin{aligned} T_3^{\alpha,\varepsilon} &\leqslant C\int_0^T \int_{B_+^d} \int_{B_+^d} |u_0(y) - u_0(x)|\theta_\alpha(t)\gamma_\varepsilon(x-y) \, dy \, dx \, dt \\ &\quad + C\int_0^T \int_{B_+^d} \int_{B_+^d} |v(0,x) - v(t,x)|\theta_\alpha(t)\gamma_\varepsilon(x-y) \, dy \, dx \, dt \\ &\leqslant C(\overline{\varepsilon} + \varepsilon_d + \alpha). \end{aligned} \tag{36}$$

The fourth term $T_4^{\alpha,\varepsilon}$
We have, for all (t,x,ξ),

$$\begin{aligned} |\nabla((jf_+)^{\alpha,\varepsilon}\lambda)(t,x,\xi)| &= \left| \int_0^\infty \int_{B_+^d} j(y) f_+(s,y,\xi)\theta_\alpha(t-s)(\nabla\gamma_\varepsilon(x-y)\lambda(x) + \gamma_\varepsilon(x-y)\nabla\lambda(x)) \, dy \, ds \right| \\ &\leqslant C||\nabla\gamma_\varepsilon||_{L^1(\mathbb{R}^d)} + C||\nabla\lambda||_{L^\infty(\mathbb{R}^d)} \leqslant \frac{C}{\overline{\varepsilon}} + \frac{C}{\varepsilon_d} + C \leqslant \frac{C}{\overline{\varepsilon}} + \frac{C}{\varepsilon_d} \end{aligned}$$

(recall that $\varepsilon_d \leqslant 1$). Hence, by (17),

$$T_4^{\alpha,\varepsilon} \leqslant \eta \int_0^T \int_{B_+^d} |Z|(t,x) \left(\sup_\xi |\nabla((jf_+)^{\alpha,\varepsilon}\lambda)|(t,x,\xi) \right) dt dx \leqslant \frac{C\eta}{\overline{\varepsilon}} + \frac{C\eta}{\varepsilon_d}. \tag{37}$$

To sum up, gathering (30), (34), (35), (36) and (37), we have proved so far that

$$\begin{aligned} &\int_{B_+^d} \lambda(x)(u(T,x) - v(T,x))^+ \, dx \\ &\qquad \leqslant C\left(\overline{\varepsilon} + \varepsilon_d + \alpha + \frac{\eta}{\overline{\varepsilon}} + \frac{\eta}{\varepsilon_d}\right) + C\int_0^T \int_{B_+^d} (u(t,x) - v(t,x))^+ \, dx \, dt + T_5^{\alpha,\varepsilon}. \end{aligned} \tag{38}$$

The aim of the following section is to estimate $T_5^{\alpha,\varepsilon}$. Using boundary layers arguments (see the introduction), we give in subsection 8.1 of the appendix an insight of the reason why this term can be bounded. However, this is only an insight: since we also want to consider irregular solutions to (1), we cannot in general estimate $T_5^{\alpha,\varepsilon}$ using boundary layers analysis.

6 Estimate for the boundary term

This estimate is made in several steps. First, using the BLN condition, we introduce f_+^b and give an upper bound $\overline{T_5}^{\alpha,\varepsilon}$ to $T_5^{\alpha,\varepsilon}$. Then, we want to see $(Ha)_d$ in $\overline{T_5}^{\alpha,\varepsilon}$, in order to express $\overline{T_5}^{\alpha,\varepsilon}$ as a part of the interior term in (16); to this end, we use Lemma 1. Finally, we must regularize the function f_+^b introduced above in order that $\overline{T_5}^{\alpha,\varepsilon}$ appears in (16) for some *regular* Ψ. The resulting term $S^{\alpha,\varepsilon}$ is then estimated.

6.1 Introduction of f_+^b

We have

$$T_5^{\alpha,\varepsilon} = \int_0^\infty \int_{B^{d-1}} \int_{\mathbb{R}_\xi} l(\overline{y})(-a(\xi)\cdot n(\overline{y}))f_+^\tau(s,\overline{y},\xi)\Psi(s,\overline{y},\xi)\,d\xi\,d\overline{y}\,ds$$

where

$$\Psi(s,\overline{y},\xi) = \int_0^T \int_{B_+^d} \theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\theta_{\varepsilon_d}(x_d)(-j(x)g_-(t,x,\xi))\lambda(x)\,dx\,dt.$$

As $(-g_-)$, Ψ is non-negative and non-decreasing with respect to ξ. Thus, (18) implies

$$\begin{aligned} T_5^{\alpha,\varepsilon} &\leqslant \overline{T_5}^{\alpha,\varepsilon} := \int_0^\infty \int_{B^{d-1}} \int_{\mathbb{R}_\xi} l(\overline{y})(-a(\xi)\cdot n(\overline{y}))f_+^\tau(s,\overline{y},\xi)f_+^b(s,\overline{y},\xi)\Psi(s,\overline{y},\xi)\,d\xi\,d\overline{y}\,ds \qquad (39)\\ &= \int_0^T \int_{B_+^d} \int_{\mathbb{R}_\xi} j(x)g_-(t,x,\xi)\Phi_0(t,x,\xi)\,d\xi\,dt\,dx \end{aligned}$$

with

$$\Phi_0(t,x,\xi) = \theta_{\varepsilon_d}(x_d)\int_0^\infty \int_{B^{d-1}} \lambda(x)l(\overline{y})(a(\xi)\cdot n(\overline{y}))f_+^\tau(s,\overline{y},\xi)f_+^b(s,\overline{y},\xi)\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds.$$

6.2 Apparition of Ha

By Lemma 1, we have $l(\overline{y})(a(\xi)\cdot n(\overline{y})) = -j(\overline{y})(H(\overline{y})a(\xi))_d$. Thus, if we define

$$\Phi(t,x,\xi) = \theta_{\varepsilon_d}(x_d)(-(H(x)a(\xi))_d)\int_0^\infty \int_{B^{d-1}} \lambda(\overline{y})j(\overline{y})f_+^\tau(s,\overline{y},\xi)f_+^b(s,\overline{y},\xi)\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds,$$

we have

$$\begin{aligned} &|\Phi_0(t,x,\xi)-\Phi(t,x,\xi)|\\ &\quad\leqslant \int_0^\infty \int_{B^{d-1}} j(\overline{y})|(H(\overline{y})a(\xi))_d\lambda(x)-(H(x)a(\xi))_d\lambda(\overline{y})|\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds\,\theta_{\varepsilon_d}(x_d)\\ &\quad\leqslant C(\overline{\varepsilon}+\varepsilon_d)\theta_{\varepsilon_d}(x_d) \end{aligned}$$

(we used the fact that H and λ are Lipschitz continuous) and

$$\overline{T_5}^{\alpha,\varepsilon} \leqslant \int_0^T \int_{B_+^d} \int_{\mathbb{R}_\xi} j(x)g_-(t,x,\xi)\Phi(t,x,\xi)\,d\xi\,dx\,dt + C(\overline{\varepsilon}+\varepsilon_d). \qquad (40)$$

6.3 Regularization of f_+^b

We now want to replace $f_+^b(s,\overline{y},\xi)$ in Φ by a regular approximation. Let $\mathrm{sgn}_{+,\delta}$ be a regular non-decreasing function, equal to 0 on $\mathbb{R}^-$, to 1 on $[\delta,\infty[$, such that $|(\mathrm{sgn}_{+,\delta})'|\leqslant C/\delta$ and $\mathrm{sgn}_{+,\delta}\to\mathrm{sgn}_+$ everywhere as $\delta\to 0$. We have, for all $(a,b,\xi)\in\mathbb{R}^3$,

$$\int_{\mathbb{R}_\xi}|\mathrm{sgn}_{+,\delta}(a-\xi)-\mathrm{sgn}_+(b-\xi)|\,d\xi\leqslant|a-b|+\delta. \tag{41}$$

Thus, defining

$$\begin{aligned}\Phi_\delta(t,x,\xi) &= \theta_{\varepsilon_d}(x_d)(-(H(x)a(\xi))_d)\\ &\quad\times\int_0^\infty\int_{B^{d-1}}\lambda(\overline{y})j(\overline{y})f_+^\tau(s,\overline{y},\xi)\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds,\end{aligned}$$

we have

$$\begin{aligned}&|\Phi(t,x,\xi)-\Phi_\delta(t,x,\xi)|\\ &\leqslant C\theta_{\varepsilon_d}(x_d)\int_0^\infty\int_{B^{d-1}}|\mathrm{sgn}_+(u_b(s,\overline{y})-\xi)-\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)|\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds\end{aligned}$$

and therefore, by (41),

$$\begin{aligned}&\left|\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi}j(x)g_-(t,x,\xi)\Phi(t,x,\xi)\,d\xi\,dx\,dt-\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi}j(x)g_-(t,x,\xi)\Phi_\delta(t,x,\xi)\,d\xi\,dx\,dt\right|\\ &\leqslant C\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi}\int_0^\infty\int_{B^{d-1}}|\mathrm{sgn}_+(u_b(s,\overline{y})-\xi)-\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)|\\ &\qquad\qquad\times\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\theta_{\varepsilon_d}(x_d)\,d\overline{y}\,ds\,d\xi\,dx\,dt\\ &\leqslant C\int_0^T\int_{B_+^d}\int_0^\infty\int_{B^{d-1}}(|u_b(s,\overline{y})-u_b(t,\overline{x})|+\delta)\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds\,\theta_{\varepsilon_d}(x_d)\,dx\,dt\\ &\leqslant C(\overline{\varepsilon}+\alpha+\delta)\int_0^T\int_{B_+^d}\int_0^\infty\int_{B^{d-1}}\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\theta_{\varepsilon_d}(x_d)\,d\overline{y}\,ds\,dx\,dt\\ &\leqslant C(\overline{\varepsilon}+\alpha+\delta)\end{aligned}$$

(we used the fact that u_b is Lipschitz continuous). We deduce from this last inequality and (40) that

$$\begin{aligned}&\overline{T_5}^{\alpha,\varepsilon}\\ &\leqslant\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi}j(x)g_-(t,x,\xi)\Phi_\delta(t,x,\xi)\,d\xi\,dx\,dt+C(\overline{\varepsilon}+\varepsilon_d+\alpha+\delta)\\ &\leqslant\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi}j(x)g_-(t,x,\xi)(H(x)a(\xi))_d\\ &\quad\times\left[\int_0^\infty\int_{B^{d-1}}\lambda(\overline{y})j(\overline{y})f_+^\tau(s,\overline{y},\xi)\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds\right]\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)(-\theta_{\varepsilon_d}(x_d))\,d\xi\,dx\,dt\\ &\quad+C(\overline{\varepsilon}+\varepsilon_d+\alpha+\delta)\end{aligned}$$

Let $\Theta_{\varepsilon_d}(x_d)=\int_0^{x_d}\theta_{\varepsilon_d}(r)\,dr$ (we have $0\leqslant\Theta_{\varepsilon_d}\leqslant 1$ and $\Theta_{\varepsilon_d}=1$ on $[\varepsilon_d,+\infty[$),

$$\Gamma(t,x,\xi)=\left[\int_0^\infty\int_{B^{d-1}}\lambda(\overline{y})j(\overline{y})f_+^\tau(s,\overline{y},\xi)\theta_\alpha(t-s)\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})\,d\overline{y}\,ds\right]\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)(1-\Theta_{\varepsilon_d}(x_d))$$

and

$$S^{\alpha,\varepsilon}=\int_0^T\int_{B^d_+}\int_{\mathbb{R}_\xi} j(x)g_-(t,x,\xi)(H(x)a(\xi))_d\partial_{x_d}\Gamma(t,x,\xi)\,d\xi\,dx\,dt.$$

The last estimate on $\overline{T_5}^{\alpha,\varepsilon}$ can be re-written

$$\overline{T_5}^{\alpha,\varepsilon}\leqslant S^{\alpha,\varepsilon}+C(\overline{\varepsilon}+\varepsilon_d+\alpha+\delta). \tag{42}$$

6.4 Estimate of $S^{\alpha,\varepsilon}$ and conclusion concerning the boundary term

The functions $f^\tau_+(s,\overline{y},\xi)$ and $\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)$ are non-negative and non-increasing with respect to ξ. Since $1-\Theta_{\varepsilon_d}\geqslant 0$, Γ is non-increasing with respect to ξ; it is also regular in (t,x). Moreover, we can see that $t\to\int_{B^d_+}\int_{\mathbb{R}_\xi}(jg_-)(t,x,\xi)\Gamma(t,x,\xi)\,d\xi\,dx$ is continuous (this is slightly more difficult to write than the continuity of (12), but similar). Hence, using $\Gamma(t,x,\xi)w_\beta(t)$ as a test function in (16), where $w_\beta(t)=\int_{t-T}^\infty\theta_\beta(s)\,ds$, and letting $\beta\to 0$ (then w_β converges to the characteristic function of $]-\infty,T]$ and w'_β converges to $-\delta_T$), we find

$$\begin{aligned}\int_0^T\int_{B^d_+}\int_{\mathbb{R}_\xi} jg_-(\partial_t\Gamma+(Ha)_{1\ldots d-1}\cdot\nabla_{\overline{x}}\Gamma+(Ha)_d\partial_{x_d}\Gamma)+\int_{B^d_+}\int_{\mathbb{R}_\xi} jf^0_-\Gamma^{(t=0)}\\ -\int_{B^d_+}\int_{\mathbb{R}_\xi}(jg_-)^{(t=T)}\Gamma^{(t=T)}+\int_0^T\int_{B^{d-1}}\int_{\mathbb{R}_\xi} l(-a\cdot n)f^b_-\overline{\Gamma}+\int_0^T\int_{B^{d-1}}\int_{\mathbb{R}_\xi} lD\delta_{u_b}\overline{\Gamma}\\ +\eta\int_0^T\int_{B^d_+}\int_{\mathbb{R}_\xi} Z\delta_v\cdot\nabla\Gamma\leqslant 0,\end{aligned}\tag{43}$$

where we have denoted $(Ha)_{1\ldots d-1}$ the vector of $\mathbb{R}^{d-1}$ made of the $d-1$ first coordinates of Ha. But, since $\theta_\alpha(-s)=0$ for $s\geqslant 0$, we have $\Gamma^{(t=0)}=0$. Moreover, $f^b_-(t,\overline{x},\xi)\mathrm{sgn}_{+,\delta}(u_b(t,\overline{x})-\xi)=0$, so that $f^b_-\overline{\Gamma}\equiv 0$. We also have

$$\int_{\mathbb{R}_\xi}\delta_{u_b(t,\overline{x})}\overline{\Gamma}(t,\overline{x},\xi)=\Gamma(t,\overline{x},0,u_b(t,\overline{x}))=0.$$

Hence, in (43), the second, fourth and fifth terms are null and we deduce

$$\begin{aligned}S^{\alpha,\varepsilon}\quad&\leqslant\quad-\int_0^T\int_{B^d_+}\int_{\mathbb{R}_\xi} jg_-(\partial_t\Gamma+(Ha)_{1\ldots d-1}\cdot\nabla_{\overline{x}}\Gamma)+\int_{B^d_+}\int_{\mathbb{R}_\xi}(jg_-)^{(t=T)}\Gamma^{(t=T)}\\ &\qquad-\eta\int_0^T\int_{B^d_+}\int_{\mathbb{R}_\xi} Z\delta_v\cdot\nabla\Gamma.\end{aligned}\tag{44}$$

We have $\Gamma\geqslant 0$ and $g_-\leqslant 0$, so that

$$\int_{B^d_+}\int_{\mathbb{R}_\xi}(jg_-)^{(t=T)}\Gamma^{(t=T)}\leqslant 0. \tag{45}$$

We have

$$\begin{aligned}|\partial_t\Gamma(t,x,\xi)|\quad&\leqslant\quad C\left(\int_0^\infty|\theta'_\alpha(t-s)|\,ds+(\mathrm{sgn}_{+,\delta})'(u_b(t,\overline{x})-\xi)|\partial_t u_b(t,\overline{x})|\right)(1-\Theta_{\varepsilon_d}(x_d))\\ &\leqslant\quad\left(\frac{C}{\alpha}+C(\mathrm{sgn}_{+,\delta})'(u_b(t,\overline{x})-\xi)\right)(1-\Theta_{\varepsilon_d}(x_d)).\end{aligned}$$

Since $\int_{\mathbb{R}_\xi}(\mathrm{sgn}_{+,\delta})'(a-\xi)\,d\xi = 1$ for all $a\in\mathbb{R}$, this implies

$$\begin{aligned}
-\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} jg_-\partial_t\Gamma &\leqslant \int_0^T\int_{B_+^d}\left(\frac{C}{\alpha}+C\int_{\mathbb{R}_\xi}(\mathrm{sgn}_{+,\delta})'(u_b(t,\overline{x})-\xi)\,d\xi\right)(1-\Theta_{\varepsilon_d}(x_d))\,dxdt \\
&\leqslant \left(\frac{C\varepsilon_d}{\alpha}+C\varepsilon_d\right)
\end{aligned} \tag{46}$$

(indeed, $\int_0^\infty(1-\Theta_{\varepsilon_d}(x_d))\,dx_d\leqslant\varepsilon_d$ since $(1-\Theta_{\varepsilon_d}(x_d))=0$ for $x_d\geqslant\varepsilon_d$ and $0\leqslant 1-\Theta_{\varepsilon_d}\leqslant 1$).
In the same way,

$$\begin{aligned}
|\nabla_{\overline{x}}\Gamma(t,x,\xi)| &\leqslant C\left(\int_{B^{d-1}}|\nabla_{\overline{x}}\overline{\gamma}_{\overline{\varepsilon}}(\overline{x}-\overline{y})|\,d\overline{y}+(\mathrm{sgn}_{+,\delta})'(u_b(t,\overline{x})-\xi)|\nabla_{\overline{x}}u_b(t,\overline{x})|\right)(1-\Theta_{\varepsilon_d}(x_d)) \\
&\leqslant \left(\frac{C}{\overline{\varepsilon}}+C(\mathrm{sgn}_{+,\delta})'(u_b(t,\overline{x})-\xi)\right)(1-\Theta_{\varepsilon_d}(x_d)),
\end{aligned} \tag{47}$$

and

$$-\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} jg_-(Ha)_{1\ldots d-1}\cdot\nabla_{\overline{x}}\Gamma\leqslant\left(\frac{C\varepsilon_d}{\overline{\varepsilon}}+C\varepsilon_d\right). \tag{48}$$

Inequality (47) and the definition of $\mathrm{sgn}_{+,\delta}$ shows that, for all (t,x,ξ),

$$|\nabla_{\overline{x}}\Gamma(t,x,\xi)|\leqslant\left(\frac{C}{\overline{\varepsilon}}+\frac{C}{\delta}\right).$$

Moreover,

$$|\partial_{x_d}\Gamma(t,x,\xi)|\leqslant C\theta_{\varepsilon_d}(x_d)\leqslant\frac{C}{\varepsilon_d}.$$

Hence, for all (t,x,ξ), $|\nabla\Gamma(t,x,\xi)|\leqslant\frac{C}{\overline{\varepsilon}}+\frac{C}{\delta}+\frac{C}{\varepsilon_d}$ and, Z being bounded in $\mathsf{L}^1((0,T)\times B_+^d)$,

$$-\eta\int_0^T\int_{B_+^d}\int_{\mathbb{R}_\xi} Z\delta_v\cdot\nabla\Gamma\leq\frac{C\eta}{\overline{\varepsilon}}+\frac{C\eta}{\delta}+\frac{C\eta}{\varepsilon_d}. \tag{49}$$

Gathering (45), (46), (48) and (49) in (44), we obtain

$$S^{\alpha,\varepsilon}\leq C\left(\varepsilon_d+\frac{\varepsilon_d}{\alpha}+\frac{\varepsilon_d}{\overline{\varepsilon}}+\frac{\eta}{\overline{\varepsilon}}+\frac{\eta}{\delta}+\frac{\eta}{\varepsilon_d}\right).$$

which gives, thanks to (39) and (42),

$$T_5^{\alpha,\varepsilon}\leq C\left(\overline{\varepsilon}+\varepsilon_d+\alpha+\delta+\frac{\varepsilon_d}{\alpha}+\frac{\varepsilon_d}{\overline{\varepsilon}}+\frac{\eta}{\overline{\varepsilon}}+\frac{\eta}{\delta}+\frac{\eta}{\varepsilon_d}\right). \tag{50}$$

7 Conclusion

We now sum up and conclude.

Combining (38) and (50) (recall that the estimates in Sections 5 and 6 concern, in fact, $\tilde{u}$ and $\tilde{v}$ — *i.e.* u and v transported), we find

$$\begin{aligned}
\int_{B_+^d}\tilde{\lambda}(x)(\tilde{u}(T,x)-\tilde{v}(T,x))^+\,dx &\leqslant C\left(\overline{\varepsilon}+\varepsilon_d+\alpha+\delta+\frac{\varepsilon_d}{\alpha}+\frac{\varepsilon_d}{\overline{\varepsilon}}+\frac{\eta}{\overline{\varepsilon}}+\frac{\eta}{\varepsilon_d}+\frac{\eta}{\delta}\right) \\
&\quad +C\int_0^T\int_{B_+^d}(\tilde{u}(t,x)-\tilde{v}(t,x))^+\,dx\,dt.
\end{aligned}$$

Minimizing on δ, α, $\overline{\varepsilon}$ and ε_d, we notice that an optimal choice of these parameters is $\delta = \eta^{1/2}$, $\varepsilon_d = \eta^{2/3}$, $\alpha = \overline{\varepsilon} = \eta^{1/3}$; we then re-transport this estimate on $\Omega \cap O$:

$$\int_{\Omega\cap O} \lambda(x)(u(T,x) - v(T,x))^+ \, dx \leqslant C\eta^{1/3} + C \int_0^T \int_{\Omega\cap O} (u(t,x) - v(t,x))^+ \, dx \, dt.$$

Summing on the local charts (recall that in the preceding inequality $\lambda = \lambda_i$ and $O = O_i$ for any $i \in \{1, \dots, n\}$) and using (14), we deduce

$$\int_{\Omega} (u(T,x) - v(T,x))^+ \, dx \leqslant C\eta^{1/3} + C \int_0^T \int_{\Omega} (u(t,x) - v(t,x))^+ \, dx \, dt.$$

This inequality being true for all $T \in [0, T_0]$, Gronwall's lemma applied to the continuous function $T \to \int_\Omega (u(T,x) - v(T,x))^+ \, dx$ ensures that there exists $C > 0$ such that, for all $T \in [0, T_0]$,

$$\int_{\Omega} (u(T,x) - v(T,x))^+ \, dx \leqslant C\eta^{1/3}. \tag{51}$$

Now, since u satisfies (5)—(6) for f_-, we see that $-u$ satisfies these equations for f_+ with $-u_0$, $-u_b$, s_*m and $s_*m^b_-$ instead of u_0, u_b, m and m^b_+ (where s is the symmetry with respect to ξ). Similarly, $-v$ satisfies (7) for g_- with $-u_0$, $-u_b$ and s_*q instead of u_0, u_b and q. Hence, (51) applied to $-u$ and $-v$ gives

$$\int_{\Omega} (-u(T,x) + v(T,x))^+ \, dx = \int_{\Omega} (u(T,x) - v(T,x))^- \, dx \leqslant C\eta^{1/3}.$$

which concludes the proof of Theorem 1.

8 Appendix

8.1 Estimate of $T_5^{\alpha,\varepsilon}$ using boundary layers

If the solution u is regular, then $T_5^{\alpha,\varepsilon}$ can be estimated using boundary layer techniques. This is what we briefly explain here.

To simplify the exposition, we take $\Omega =]0, \infty[$ and recall some basic facts concerning boundary layers: if u is regular, then the parabolic approximation admits the decomposition $v(t,x) = u(t,x) + c(t, x/\eta^\gamma) + r^\eta(t,x)$, where $\gamma = 1/2$ or 1 depending if the boundary is characteristic or not, and r^η is a remainder (small, with respect to η, in L^∞ norm). Fix $t \in (0,T)$, set $w(y) = u(t,0) + c(t,y)$, $w_0 = u_b(t)$ and $w_\infty = u(t,0)$. Then, by properties of the layer c, w satisfies

$$\dot{w}(y) = A(w(y)) - A(w_\infty), \tag{52}$$
$$w(0) = w_0, \tag{53}$$
$$w(+\infty) = w_\infty. \tag{54}$$

Notice that, since (52) is an autonomous o.d.e., $\dot{w}$ vanishes on $[0, +\infty)$ if, and only if, w is constant (and then $w_0 = w_\infty$). Now, suppose $w_0 \neq w_\infty$. Then, since $\dot{w}$ does not vanish, it has a constant sign, which is actually the sign of $w_\infty - w_0$ since w is an orbit from w_0 to w_∞. To sum up, we have $\mathrm{sgn}(w_\infty - w_0)\dot{w}(y) \geq 0$ for all $y \geq 0$. In view of (52), this is equivalent to $\mathrm{sgn}(w_\infty - w_0)(A(w(y)) - A(w_\infty)) \geq 0$ for all $y \geq 0$ or still, since w is a bijection $[0, +\infty) \to [w_0, w_\infty)$,

$$\forall \kappa \in [w_0, w_\infty]\,, \ \mathrm{sgn}(w_\infty - w_0)(A(\kappa) - A(w_\infty)) \geq 0\,. \tag{55}$$

Conversely, on can check that (55) is a sufficient condition to the existence of a solution to (52)-(53)-(54). Now, replacing w_0 and w_∞ by their respective values $u_b(t)$ and $u(t,0)$, (55) appears to be nothing but the BLN condition

$$\forall k \in [u_b(t), u(t,0)]\,,\ -\mathrm{sgn}(u(t,0) - u_b(t))(A(u(t,0)) - A(k)) \geq 0\,.$$

In other words, the BLN condition is a necessary and sufficient condition to the existence of the boundary layer function c.

Let us now come back to the estimate of $T_5^{\alpha,\varepsilon}$: assuming that $\alpha = 0$ and $\lambda \equiv 1$, $T_5^{\alpha,\varepsilon}$ reduces in this setting to

$$\begin{aligned}\tilde{T}_5^\varepsilon &= \int_0^T \int_0^\infty \int_{\mathbb{R}_\xi} a(\xi)\mathrm{sgn}_+(u(t,0) - \xi)(-\mathrm{sgn}_-(v(t,x) - \xi))\theta_\varepsilon(x)\, d\xi\, dx\, dt \\ &= \int_0^T \int_0^\infty \mathrm{sgn}_+(u(t,0) - v(t,x))(A(u(t,0)) - A(v(t,x)))\theta_\varepsilon(x)\, dx\, dt.\end{aligned}$$

Since $\zeta \to \mathrm{sgn}_+(u(t,0) - \zeta)(A(u(t,0)) - A(\zeta))$ is Lipschitz continuous, $\tilde{T}_5^\varepsilon$ can be assimilated, up to an error of order $\eta + \varepsilon$, to

$$\int_0^T \int_0^\infty \mathrm{sgn}_+(-c(t, x/\eta^\gamma))(A(u(t,0)) - A(u(t,0) + c(t, x/\eta^\gamma))\theta_\varepsilon(x)\, dx\, dt\,.$$

Since w is monotonous between w_0 and w_∞, $w_\infty - w(y)$ has the same sign than $w_\infty - w_0$. Reporting this result in (55) and replacing w, w_0 and w_∞ by $u(t,0) + c(t,y)$, $u_b(t)$ and $u(t,0)$ respectively we get

$$\mathrm{sgn}(-c(t,y))(A(u(t,0)) - A(u(t,0) + c(t,y)) \leq 0$$

for all $y \geq 0$, which shows that, up to an error of order $\eta + \varepsilon$, $T_5^{\alpha,\varepsilon}$ is nonpositive.

The basic idea in Section 6 is thus to compare $T_5^{\alpha,\varepsilon}$ to some nonpositive quantity, which is done as early as Subsection 6.1.

8.2 Technical results

The first lemma is classical, we do not prove it.

Lemma 2 *Let U be a bounded open set of $\mathbb{R}^d$ with Lipschitz continuous boundary and γ_ε be a regularizing kernel with support contained in the ball of radius $|\varepsilon|$. Then there exists C only depending on U such that, for all $w \in \mathsf{L}^1(U) \cap \mathsf{BV}(U)$,*

$$\int_U \int_U |w(x) - w(y)|\gamma_\varepsilon(x - y)\, dx\, dy \leqslant C|\varepsilon|(||w||_{\mathsf{L}^1(U)} + |w|_{\mathsf{BV}(U)}).$$

The second lemma is a technical result used in Section 5.

Lemma 3 *Let $D > 0$ and K be a compact subset of B^d. We take $\nu \in \mathbb{R}^d$ such that $|\nu| < \mathrm{dist}(K, \mathbb{R}^d \backslash B^d)$ and j a regular function on B^d. If $w \in \mathsf{BV}(B^d_+)$ then there exists C not depending on ν or w such that*

$$\int_K \int_{-D}^D |\nabla(j(x)\mathrm{sgn}_-(w(x) - \xi))^\nu|\, d\xi\, dx \leqslant C(1 + |w|_{\mathsf{BV}(B^d_+)}).$$

Proof of Lemma 3

The proof is made in several steps. Let U be an open set relatively compact in B^d, containing K and such that $|\nu| \leqslant \mathrm{dist}(U, \mathbb{R}^d \backslash B^d)$. We prove the result of the lemma with U instead of K (the introduction of this open set is useful because we use classical results concerning BV functions on *open* sets).

Step 0: (a preliminary result) Let $r \in \mathsf{W}^{1,1}(B^d_+)$ and denote by R the extension of r to B^d by 0 outside B^d_+. Then $R \in \mathsf{BV}(B^d)$ and $|R|_{\mathsf{BV}(B^d)} \leqslant C||r||_{\mathsf{W}^{1,1}(B^d_+)}$. To see this, take $\phi \in (\mathsf{C}^\infty_\mathsf{c}(B^d))^d$; thanks to an integrate by parts, we have

$$\int_{B^d} R\mathrm{div}(\phi) = \int_{B^d_+} r\mathrm{div}(\phi) = -\int_{B^{d-1}} r\phi_d - \int_{B^d_+} \phi \cdot \nabla r.$$

The right-hand side of this equation is bounded by $C||r||_{\mathsf{W}^{1,1}(B^d_+)}||\phi||_\infty$, which proves the result (in fact, the preceding equation computes the gradient of R).

Step 1: let $\mathrm{sgn}_{-,\delta} : \mathbb{R} \to \mathbb{R}$ be a regular nondecreasing function, equal to 0 on $\mathbb{R}^+$, to -1 on $]-\infty, -\delta]$ and such that $\mathrm{sgn}_{-,\delta} \to \mathrm{sgn}_-$ as $\delta \to 0$. We prove the result when $w \in \mathsf{W}^{1,1}(B^d_+)$ and sgn_- is replaced by $\mathrm{sgn}_{-,\delta}$, with C not depending on δ.
We clearly have (since $\mathrm{sgn}_{-,\delta}$ is regular) $j\mathrm{sgn}_{-,\delta}(w-\xi) \in \mathsf{W}^{1,1}(B^d_+)$ and

$$\nabla(j\mathrm{sgn}_{-,\delta}(w-\xi)) = \nabla j\, \mathrm{sgn}_{-,\delta}(w-\xi) + j(\mathrm{sgn}_{-,\delta})'(w-\xi)\nabla w.$$

By Step 0, the extension W_ξ of $j\mathrm{sgn}_{-,\delta}(w-\xi)$ to B^d by 0 outside B^d_+ is in $\mathsf{BV}(B^d)$ and

$$|W_\xi|_{\mathsf{BV}(B^d)} \leqslant C + C||(\mathrm{sgn}_{-,\delta})'(w-\xi)\nabla w||_{L^1(B^d_+)},$$

where C does not depend on δ nor w.
Moreover, by choice of ν, $(j(\cdot)\mathrm{sgn}_{-,\delta}(w(\cdot)-\xi))^\nu = W_\xi \star \gamma_\nu$ and $\nabla(j\mathrm{sgn}_{-,\delta}(w-\xi))^\nu = \nabla W_\xi \star \gamma_\nu$ on U. Thus,

$$||\nabla(j\mathrm{sgn}_{-,\delta}(w-\xi))^\nu||_{L^1(U)} \leqslant |W_\xi|_{\mathsf{BV}(B^d)} \leqslant C + C\int_{B^d_+} (\mathrm{sgn}_{-,\delta})'(w-\xi)|\nabla w|$$

We now integrate with respect to ξ and use $\int_{-D}^D (\mathrm{sgn}_{-,\delta})'(s-\xi)\,d\xi \leqslant \int_{\mathbb{R}_\xi} (\mathrm{sgn}_{-,\delta})'(s-\xi)\,d\xi = 1$ for all $s \in \mathbb{R}$ to find

$$\int_U \int_{-D}^D |\nabla(j\mathrm{sgn}_{-,\delta}(w-\xi))^\nu| \leqslant C + C\int_{B^d_+} |\nabla w|$$

which concludes this step.

Step 2: conclusion.
There exists $w_n \in W^{1,1}(B^d_+)$ which converge to w in $L^1(B^d_+)$ and such that $|w_n|_{\mathsf{BV}(B^d_+)} \to |w|_{\mathsf{BV}(B^d_+)}$.
Since $\mathrm{sgn}_{-,\delta}$ is regular, $j\mathrm{sgn}_{-,\delta}(w_n-\xi) \to j\mathrm{sgn}_{-,\delta}(w-\xi)$ in $\mathsf{L}^1(B^d_+)$ as $n \to \infty$ so that $(j\mathrm{sgn}_{-,\delta}(w_n-\xi))^\nu \to (j\mathrm{sgn}_{-,\delta}(w-\xi))^\nu$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Moreover, $\mathrm{sgn}_{-,\delta}(w-\xi) \to \mathrm{sgn}_-(w-\xi)$ in $L^1(B^d_+)$ as $\delta \to 0$ so that $(j\mathrm{sgn}_{-,\delta}(w-\xi))^\nu \to (j\mathrm{sgn}_-(w-\xi))^\nu$ in $\mathsf{L}^1(\mathbb{R}^d)$ as $\delta \to 0$.
We deduce that

$$\begin{aligned}
\int_U |\nabla(j\mathrm{sgn}_-(w-\xi))^\nu| &= |(j\mathrm{sgn}_-(w-\xi))^\nu|_{\mathsf{BV}(U)} \\
&\leqslant \liminf_{\delta\to 0} |(j\mathrm{sgn}_{-,\delta}(w-\xi))^\nu|_{\mathsf{BV}(U)} \\
&\leqslant \liminf_{\delta\to 0}\liminf_{n\to\infty} |(j\mathrm{sgn}_{-,\delta}(w_n-\xi))^\nu|_{\mathsf{BV}(U)} \\
&= \liminf_{\delta\to 0}\liminf_{n\to\infty} \int_U |\nabla(j\mathrm{sgn}_{-,\delta}(w_n-\xi))^\nu|.
\end{aligned}$$

Integrating on $\xi \in [-D, D]$ and using Fatou's Lemma, the result of Step 1 gives

$$\begin{aligned}
\int_U \int_{-D}^{D} |\nabla(j\mathrm{sgn}_-(w-\xi))^\nu| &\leqslant \liminf_{\delta\to 0} \liminf_{n\to\infty} \int_{-D}^{D} \int_U |\nabla(j\mathrm{sgn}_{-,\delta}(w_n-\xi))^\nu| \\
&\leqslant \liminf_{\delta\to 0} \liminf_{n\to\infty} \left(C + C \int_{B_+^d} |\nabla w_n(x)|\, dx \right) \\
&\leqslant \liminf_{\delta\to 0} (C + C|w|_{\mathsf{BV}(B_+^d)}) = C + C|w|_{\mathsf{BV}(B_+^d)}
\end{aligned}$$

by choice of $(w_n)_{n\geqslant 1}$. ■

References

[1] C. Bardos, A. Y. le Roux, and J.-C. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, 4 (1979), pp. 1017–1034.

[2] C. Chainais-Hillairet and E. Grenier, *Numerical boundary layers for hyperbolic systems in 1-D*, M2AN Math. Model. Numer. Anal., 35 (2001), pp. 91–106.

[3] M. Gisclon and D. Serre, *Conditions aux limites pour un système strictement hyperbolique fournies par le schéma de Godunov*, RAIRO Modél. Math. Anal. Numér., 31 (1997), pp. 359–380.

[4] E. Grenier and O. Guès, *Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems*, J. Differential Equations, 143 (1998), pp. 110–146.

[5] O. Guès, *Perturbations visqueuses de problèmes mixtes hyperboliques et couches limites*, Ann. Inst. Fourier (Grenoble), 45 (1995), pp. 973–1006.

[6] C. Imbert and J. Vovelle, *A kinetic formulation for multidimensional scalar conservation laws with boundary conditions and applications*, submitted.

[7] K. T. Joseph and P. G. LeFloch, *Boundary layers in weak solutions of hyperbolic conservation laws*, Arch. Ration. Mech. Anal., 147 (1999), pp. 47–88.

[8] S. N. Kružkov, *First order quasilinear equations with several independent variables.*, Mat. Sb. (N.S.), 81 (123) (1970), pp. 228–255.

[9] N. N. Kuznecov, *The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation*, Ž. Vyčisl. Mat. i Mat. Fiz., 16 (1976), pp. 1489–1502, 1627.

[10] P.-L. Lions, B. Perthame, and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc., 7 (1994), pp. 169–191.

[11] F. Otto, *Initial-boundary value problem for a scalar conservation law*, C. R. Acad. Sci. Paris Sér. I Math., 322 (1996), pp. 729–734.

[12] B. Perthame, *Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure*, J. Math. Pures Appl. (9), 77 (1998), pp. 1055–1064.

[13] E. Tadmor and T. Tang, *Pointwise error estimates for relaxation approximations to conservation laws*, SIAM J. Math. Anal., 32 (2000), pp. 870–886 (electronic).

[14] T. Tang, *Error estimates of approximate solutions for nonlinear scalar conservation laws*, in Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000), vol. 141 of Internat. Ser. Numer. Math., 140, Birkhäuser, Basel, 2001, pp. 873–882.